INSTITVTIONVM CALCVLI INTEGRALIS

VOLVMEN TERTIVM,

IN QVO METHODVS INVENIENDI FVNCTIONES DVARVM ET PLVRIVM VARIABILIVM, EX DATA RELATIONE DIFFERENTIALIVM CVIVSVIS GRADVS PERTRACTATVR.

VNA CVM APPENDICE DE CALCVLO VARIATIONVM ET SVPPLEMENTO, EVOLVTIONEM CASVVM PRORSVS SINGVLARIVM CIRCA INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM CONTINENTE.

AVCTORE

LEONHARDO EVLERO

ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO ACAD. PETROP. PARISIN. ET LONDIN.

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CALCVLI

CALCVLI INTEGRALIS
LIBER POSTERIOR.

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CAPVT I.

DE NATVRA AEQVATIONVM DIFFERENTIALIVM QVIBVS FVNCTIONES DVARVM VARIABILIVM DETERMINANTVR IN GENERE.

Problema 1.

1.

Si z sit functio quaecunque duarum variabilium x et y, definire indolem aequationis differentialis qua relatio differentialium dx, dy et dz exprimitur.

Solutio.

Sit $Pdx + Qdy + Rdz = 0$ aequatio relationem differentialium dx, dy et dz exprimens, in qua P, Q et R ſint functiones quaecunque ipſarum x, y et z. Ac primo quidem neceſſe eſt, vt haec aequatio nata ſit ex differentiatione aequationis cuiuspiam finitae poſtquam differentiale per quampiam quantitatem fuerit diuiſum. Dabitur ergo quidam multiplicator puta M, per quem formula

$$P\,dx + Q\,dy + R\,dz$$

multiplicata fiat integrabilis; niſi enim talis multiplicator exiſteret, aequatio differentialis propoſita foret abſurda, nihilque omnino declararet. Totum ergo negotium huc redit, vt character aſſignetur, cuius ope huiusmodi aequationes differentiales abſurdae nihilque ſignificantes a realibus dignoſci queant. Hunc in finem contemplemur aequationem propoſitam $P\,dx + Q\,dy + R\,dz = 0$ tanquam realem. Sit M multiplicator eam reddens integrabilem, ita vt haec formula

$$MP\,dx + MQ\,dy + MR\,dz$$

ſit verum differentiale cuiuspiam functionis trium variabilium x, y et z; quae functio ſi ponatur $= V$ haec aequatio $V = \text{Conſt.}$ futura ſit integrale completum aequationis propoſitae. Siue igitur x, ſiue y, ſiue z accipiatur conſtans, ſingulas has formulas: $MQ\,dy + MR\,dz$; $MR\,dz + MP\,dx$; $MP\,dx + MQ\,dy$;

ſeorſim integrabiles eſſe oportet; vnde ex natura differentialium erit

$$\left(\frac{d.MQ}{dz}\right) - \left(\frac{d.MR}{dy}\right) = 0; \quad \left(\frac{d.MR}{dx}\right) - \left(\frac{d.MP}{dz}\right) = 0;$$
$$\left(\frac{d.MP}{dy}\right) - \left(\frac{d.MQ}{dx}\right) = 0$$

vnde

vnde per euolutionem hae tres oriuntur aequationes:

$$\text{I.}\quad M\left(\frac{dQ}{dz}\right)+Q\left(\frac{dM}{dz}\right)-M\left(\frac{dR}{dy}\right)-R\left(\frac{dM}{dy}\right)=0$$

$$\text{II.}\quad M\left(\frac{dR}{dx}\right)+R\left(\frac{dM}{dx}\right)-M\left(\frac{dP}{dz}\right)-P\left(\frac{dM}{dz}\right)=0$$

$$\text{III.}\quad M\left(\frac{dP}{dy}\right)+P\left(\frac{dM}{dy}\right)-M\left(\frac{dQ}{dx}\right)-Q\left(\frac{dM}{dx}\right)=0$$

quarum si prima per P, secunda per Q et tertia per R multiplicetur, in summa omnia differentialia ipsius M se tollent, et reliqua aequatio per M diuisa erit:

$$P\left(\frac{dQ}{dz}\right)-P\left(\frac{dR}{dy}\right)+Q\left(\frac{dR}{dx}\right)-Q\left(\frac{dP}{dz}\right)+R\left(\frac{dP}{dy}\right)-R\left(\frac{dQ}{dx}\right)=0,$$

quae continet characterem, aequationes differentiales reales ab absurdis discernentem, et quoties inter quantitates P, Q et R haec conditio locum habet, toties aequatio differentialis proposita

$$P\,dx+Q\,dy+R\,dz=0$$

est realis. Caeterum hic meminisse oportet, huiusmodi formulam vncinulis inclusam $\left(\frac{dQ}{dz}\right)$ significare valorem $\frac{dQ}{dz}$, si in differentiatione ipsius Q sola quantitas z vt variabilis tractetur; quod idem de ceteris est tenendum, quae ergo semper ad functiones finitas reducuntur.

Coroll. 1.

2. Proposita ergo aequatione differentiali inter tres variabiles:

$$P\,dx+Q\,dy+R\,dz=0,$$

ante omnia diſpiciendum eſt, vtrum character inventus locum habeat, nec ne? priori caſu aequatio erit realis, poſteriori vero abſurda et nihil plane ſignificans, neque vnquam ad talem aequationem vllius problematis ſolutio perducere valet.

Coroll. 2.

3. Character inuentus etiam hoc modo exprimi poteſt

$$\left(\frac{PdQ-QdP}{dz}\right)+\left(\frac{QdR-RdQ}{dx}\right)+\left(\frac{RdP-PdR}{dy}\right)=0$$

quandoquidem vncinulae non quantitates finitas afficiunt, ſed ſolam differentiationem ad certam variabilem reſtringunt.

Coroll. 3.

4. Simili modo ſi aequatio haec characterem continens per PQR diuidatur, ea hanc formam induet:

$$\left(\frac{d.l\frac{Q}{P}}{Rdz}\right)+\left(\frac{d.l\frac{R}{Q}}{Pdx}\right(+\left(\frac{d.l\frac{P}{R}}{Qdy}\right)0$$

quae etiam ita exprimi poteſt:

$$\left(\frac{\frac{dQ}{Q}-\frac{dP}{P}}{Rdz}\right)+\left(\frac{\frac{dR}{R}-\frac{dQ}{Q}}{Pdx}\right)+\left(\frac{\frac{dP}{P}-\frac{dR}{R}}{Qdy}\right)=0.$$

Scho-

Scholion 1.

5. Quemadmodum omnes aequationes differentiales inter binas variabiles semper sunt reales, semperque per eas relatio certa inter ipsas variabiles definitur, ita hinc discimus, rem secus se habere in aequationibus differentialibus, quae tres variabiles inuoluant, atque huiusmodi aequationes

$$P\,dx + Q\,dy + R\,dz = 0$$

non certam relationem inter ipsas quantitates finitas x, y et z declarare, nisi quantitates P, Q, R ita fuerint comparatae, vt character inuentus locum habeat. Ex quo intelligitur infinitas huiusmodi aequationes differentiales inter ternas variabiles proponi posse, quibus nulla prorsus relatio finita conueniat, et quae propterea nihil plane definiant. Pro arbitrio scilicet huiusmodi aequationes formari possunt, nullo scopo proposito ad quem sint accommodatae; statim enim ac certum quoddam problema ad aequationem differentialem inter ternas variabiles perducit, semper necesse est characterem assignatum ei conuenire, cum alioquin nihil omnino significaret. Talis aequatio nihil significans est exempli gratia $z\,dx + x\,dy + y\,dz = 0$, neque pro z vlla quidem functio ipsarum x et y cogitari potest quae isti aequationi satisfaciat; quin etiam character noster pro hoc exemplo dat $-x-y-z$, quae quantitas cum non euanescat, absurditatem illius aequationis declarat.

Scho-

Scholion 2.

6. Quo character inuentus facilius ad quosuis casus oblatos accommodari queat, ex aequatione

$$P\,dx + Q\,dy + R\,dz = 0$$

primo euoluantur sequentes valores:

$$\left(\frac{dQ}{dz}\right) - \left(\frac{dR}{dy}\right) = L;\ \left(\frac{dR}{dx}\right)\ \left(\frac{dP}{dz}\right) = M;\ \left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dx}\right) = N$$

et character noster hac continebitur expressione:

$$LP + MQ + NR$$

quae si euanescat, aequatio proposita erit realis, et aequationem quandam finitam agnoscet; sin autem ea ad nihilum non redigatur, aequatio proposita erit absurda, atque de eius integratione ne cogitandum quidem erit. Ita in exemplo supra posito erit

$$P = z;\ \ Q = x;\ \ R = y,$$

hinc

$$L = -1;\ M = -1 \text{ et } N = -1,$$

vnde character $-x - y - z$, absurditatem indicat. Proferamus vero etiam exemplum aequationis realis:

$$dx(yy + nyz + zz) - x(y + nz)dy - xzdz = 0$$

in qua ob

$$P = yy + nyz + zz;\ Q = -xy - nxz;\ \text{et } R = -xz$$

erit

$$L = -nx;\ M = -3z - ny \text{ et } N = 3y + 2nz$$

vnde

vnde

$$LP+MQ+NR=-nx(yy+nyz+zz)+x(y+nz)(3z+ny)$$
$$-xz(3y+2nz)=x(-nyy-nnyz-nzz+3yz+3nzz+nyy+nnyz$$
$$-3yz-2nzz)=0$$

quare cum hic character euanefcat, aequatio haec differentialis pro reali eft habenda. Simili modo propofita hac aequatione:

$$2dx(y+z)+dy(x+3y+2z)+dz(x+y)=0 \text{ ob}$$
$$P=2y+2z;\ Q=x+3y+2z;\ R=x+y \text{ fit}$$
$$L=2-1=1;\ M=1-2=-1 \text{ et } N=2-1=1,$$

hincque

$$LP+MQ+NR=2y+2z-x-3y-2z+x+y=0$$

vnde ifta aequatio differentialis erit realis.

Problema 2.

7. Propofita aequatione differentiali inter ternas variabiles x, y, z, quae fit realis, eius integrale inueftigare, vnde pateat, qualis functio vna earum fit binarum reliquarum.

Solutio.

Sit aequatio differentialis propofita:

$$Pdx+Qdy+Rdz=0$$

in qua P, Q, R eiusmodi fint functiones ipfarum x, y et z, vt character realitatis ante inuentus fatisfaciat. Nisi enim ista aequatio esset realis, ridiculum foret, eius integrationem tentare. Sumamus ergo hanc aequationem esse realem, atque dabitur relatio inter ipsas quantitates x, y et z, aequationi propositae satisfaciens; ad quam inueniendam perpendatur, si in aequatione integrali vna variabilium puta z, constans spectetur ex eius differentiali nihilo aequali posito nasci debere aequationem

$$P\,dx + Q\,dy = 0.$$

Vicissim ergo vna variabili puta z vt constante tractata, integratio aequationis differentialis

$$P\,dx + Q\,dy = 0$$

quae duas tantum variabiles continet, perducet ad aequationem integralem quaesitam, si modo in quantitatem constantem per integrationem ingressam illa quantitas z rite inuoluatur. Ex quo hanc regulam pro integratione aequationis propositae colligimus. Consideretur vna variabilium puta z vt constans, vt habeatur haec aequatio $P\,dx + Q\,dy = 0$ duas tantum variabiles x et y implicans; tum eius inuestigetur aequatio integralis completa, quae ergo constantem arbitrariam C complectetur. Deinde haec constans C consideretur vt functio quaecunque ipsius z, atque hac z nunc etiam pro variabili habita, aequatio integralis inuenta denuo differentietur; vt omnes tres x, y et z tanquam variabiles tractentur, et aequatio differentialis resultans comparetur cum proposita $P\,dx + Q\,dy + R\,dz = 0$, vbi quidem functiones

ctiones P et Q sponte prodibunt, at functio R cum ea quantitate, qua elementum dz afficitur, collata determinabit rationem, qua quantitas z in illam litteram C ingreditur, sicque obtinebitur aequatio integralis quaesita, quae simul erit completa, cum semper in illa litterae C pars quaedam constans vere arbitraria relinquatur, cum haec determinatio ex differentiali ipsius C sit petenda.

Coroll. 1.

8. Reducitur ergo integratio huiusmodi aequationum differentialium tres variabiles continentium ad integrationem aequationum differentialium inter duas tantum variabiles, quae ergo quoties licet per methodos in superiori libro traditas, est instituenda.

Coroll. 2.

9. Haec ergo integratio tribus modis institui potest prout primo vel z vel y vel x tanquam constans spectatur. Semper autem necesse est, vt eadem aequatio integralis resultet, siquidem aequatio differentialis fuerit realis.

Coroll. 3.

10. Quodsi haec methodus tentetur in aequatione differentiali impossibili, determinatio illius constantis C non ita succedet, vt eam variabilem, quae pro constante est habita, solam inuoluat; atque etiam ex hoc criterium realitatis peti poterit.

Scholion.

11. Quo haec operatio facilius intelligatur, periculum faciamus primo in aequatione impossibili hac

$$z\,dx + x\,dy + y\,dz = 0$$

hic sumta z pro constante erit

$$z\,dx + x\,dy = 0 \text{ seu } \frac{z\,dx}{x} + dy = 0$$

cuius integrale est $zlx + y = C$ existente C functione ipsius z. Differentietur ergo haec aequatio sumendo etiam z variabile, positoque $dC = D\,dz$, vt D sit etiam functio ipsius z tantum erit:

$$\frac{z\,dx}{x} + dy + dz\,lx = D\,dz \text{ seu}$$

$$z\,dx + x\,dy + dz(xlx - Dx) = 0$$

deberet ergo esse $xlx - Dx = y$ seu $D = lx - \frac{y}{x}$, quod est absurdum.

Deinde in aequatione reali

$$2\,dx(y+z) + dy(x+3y+2z) + dz(x+y) = 0$$

operatio exposita ita instituatur. Sumatur y constans vt sit

$$2\,dx(y+z) + dz(x+y) = 0 \text{ seu } \frac{2\,dx}{x+y} + \frac{dz}{y+z} = 0$$

cuius integrale est

$$2l(x+y) + l(y+z) = C,$$

vbi C etiam y inuoluat. Sit ergo $dC = D\,dy$, et sumto

ſumto etiam y variabili, differentiatio praebet

$$\frac{2dx+2dy}{x+y}+\frac{dy+dz}{y+z}=D\,dy \quad \text{ſeu}$$

$$2dx(y+z)+2dy(y+z)+dy(x+y)+dz(x+y)$$
$$=Ddy(x+y)(y+z)$$

quae expreſſio cum forma propoſita collata praebet $D=0$, ideoque $dC=0$ et C fit conſtans vera; ita vt integrale ſit

$$(x+y)^2(y+z)=\text{Conſt.}$$

Huiusmodi igitur exempla aliquot euoluamus.

Exemplum 1.

12. *Huius aequationis differentialis realis*

$$dx(y+z)+dy(x+z)+dz(x+y)=0$$

integrale inueſtigare.

Primo quidem patet hanc aequationem eſſe realem cum ſit

$$P=y+z;\quad L=1-1=0$$
$$Q=x+z;\quad M=1-1=0$$
$$R=x+y;\quad N=1-1=0$$

ſumatur igitur z conſtans, et aequatio prodibit

$$dx(y+z)+dy(x+z)=0 \quad \text{ſeu} \quad \frac{dx}{x+z}+\frac{dy}{y+z}=0$$

cuius integrale eſt

$$l(x+z)+l(y+z)=f:z,$$

 ſtatua-

ſtatuatur ergo

$$(x+z)(y+z)=Z,$$

vbi natura functionis Z ex differentiatione debet erui. Fit autem

$$dx(y+z)+dy(x+z)+dz(x+y+2z)=dZ$$

a qua ſi propoſita auferatur, relinquitur $2zdz=dZ$ hinc $Z=zz+C$, ita vt aequatio integralis completa ſit

$$(x+z)(y+z)=zz+C \text{ ſeu } xy+xz+yz=C$$

quae quidem ex ipſa propoſita

$$ydx+zdx+xdy+zdy+xdz+ydz=0$$

facile elicitur, cum bina membra iuncta ſit integrabilia.

Exemplum 2.

13. *Huius differentialis aequationis realis*

$$dx(ay-bz)+dy(cz-ax)+dz(bx-cy)=0$$

aequationem integralem completam inuenire.

Realitas huius aequationis ita oſtenditur: Cum ſit $P=ay-bz$; erit $L=2c$

$Q=cz-ax$; $M=2b$

$R=bx-cy$; $N=2a$

hincque manifeſto $LP+MQ+NR=0$.

Iam

Iam ſumatur z conſtans, vt habeatur:

$$\frac{c\,dx}{cz-ax}+\frac{dy}{ay-bz}=0 \text{ ergo } \frac{1}{a}\,l\,\frac{ay-bz}{cz-ax}=f:z$$

ſtatuatur ergo $\frac{ay-bz}{cz-ax}=Z$, et differentiatio praebet

$$\frac{a\,dx(ay-bz)+a\,dy(cz-ax)+a\,dz(bx-cy)}{(cz-ax)^2}=dZ$$

ex cuius comparatione cum propoſita fit $dZ=0$ et $Z=C$, ita vt aequatio integralis completa ſit:

$$\frac{ay-bz}{cz-ax}=n \text{ ſeu } ay+nax=(b+nc)z.$$

Quod ſi aequatio integralis ponatur

$$Ax+By+Cz=0$$

hae conſtantes ita debent eſſe comparatae vt ſit

$$Ac+Bb+Ca=0$$

ſicque conſtans arbitraria concinnius inducitur.

Corollarium.

14. Haec ergo aequatio integrabilis redditur, ſi diuidatur per $(cz-ax)^2$, atque ob eandem rationem etiam hi diuiſores:

$$(ay-bz)^2 \text{ et } (bx-cy)^2$$

idem praeſtant. Vi enim integralis hi diuiſores conſtantem inter ſe tenent rationem. Namque ſi $\frac{ay-bz}{cz-ax}=n$ erit

$$\frac{bx-cy}{cz-ax}=\frac{-b-nc}{a} \text{ et } \frac{bx-cy}{ay-bz}=\frac{-b-nc}{na}.$$

Exem-

Exemplum 3.

15. *Huius aequationis differentialis realis:*

$$dx(yy+yz+zz)+dy(zz+xz+xx)+dy(xx+xy+yy)=0$$

aequationem integralem completam inuestigare.

Realitas huius aequationis inde patet, quod sit:

$P=yy+yz+zz$ hincque $L=2z+x-x-2y=2(z-y)$

$Q=zz+xz+xx$ $\qquad M=2x+y-y-2z=2(x-z)$

$R=xx+xy+yy$ $\qquad N=2y+z-z-2x=2(y-x)$

vnde fit:

$$LP+MQ+NR=2(z^3-y^3)+2(x^3-z^3)+2(y^3-x^3)=0.$$

Ad integrale ergo inuestigandum sumatur z constans, eritque

$$\frac{dx}{xx+xz+zz}+\frac{dy}{yy+yz+zz}=0$$

cuius integrale est

$$\frac{2}{z\sqrt{3}}\text{ Ang. tang. }\frac{x\sqrt{3}}{2z+x}+\frac{2}{z\sqrt{3}}\text{ Ang. tang. }\frac{y\sqrt{3}}{2z+y}=f:z$$

quae per collectionem horum angulorum abit in:

$$\frac{2}{z\sqrt{3}}\text{ Ang. tang. }\frac{(xz+yz+xy)\sqrt{3}}{2zz+xz+yz-xy}=f:z.$$

Statuatur ergo $\frac{xz+yz+xy}{2zz+xz+yz-xy}=Z$; haecque aequatio differentietur sumtis omnibus tribus x, y et z variabilibus, ac prodibit

$$\frac{2zdx(yy+yz+zz)+2zdy(zz+xz+xx)-2xdz(zz+yz+yy)-2ydz(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=dZ,$$

cum igitur ex aequatione proposita sit

$$dx(yy+yz+zz)+dy(zz+xz+xx)=-dz(xx+xy+yy)$$

erit

erit facta substitutione

$$\frac{-2zdz(xx+xy+yy)-2xdz(zz+yz+yy)-2ydz(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=dZ \text{ seu}$$

$$\frac{-2dz(xxz+xzz+yyz+yzz+xxy+xyy+3xyz)}{(2zz+xz+yz-xy)^2}=dZ$$

quae in hanc formam reducitur:

$$\frac{-2dz(x+y+z)(xy+xz+yz)}{(2zz+xz+yz-xy)^2}=dZ.$$

At ob $Z=\frac{xy+xz+yz}{2zz+xz+yz-xy}$ erit

$$\frac{-2ZZdz(x+y+z)}{xy+xz+yz}=dZ \text{ seu } \frac{-dZ}{ZZ}=\frac{2dz(x+y+z)}{xy+xz+yz}.$$

Necesse ergo est vt etiam $\frac{xy+xz+yz}{x+y+z}$ sit functio ipsius z tantum, quae vocetur Σ, vt sit $-\frac{dZ}{ZZ}=\frac{2dz}{\Sigma}$. Verum ex sola forma functionis Z negotium confici oportet; quod ita expediri potest. Cum sit

$$Z=\frac{xy+xz+yz}{2zz+xz+yz-xy} \text{ erit } 1+Z=\frac{2zz+2xz+2yz}{2zz+xz+yz-xz}$$

hinc $\frac{1+Z}{Z}=\frac{2z(x+y+z)}{xy+xz+yz}$, cuius valoris ope quantitates x et y ex aequatione differentiali eliduntur, fitque

$$-\frac{dZ}{ZZ}=dz.\frac{2(x+y+z)}{xy+xz+yz}=dz.\frac{1+Z}{Zz},$$

vnde

$$\frac{-dZ}{Z(1+Z)}=\frac{dz}{z}=\frac{-dZ}{Z}+\frac{dZ}{1+Z},$$

et integrando $lz=l\frac{1+Z}{Z}+la$. Ergo

$$\frac{1+Z}{Z}=\frac{z}{a} \text{ et } Z=\frac{a}{z-a}$$

ita vt aequatio integralis quaesita sit

$$\frac{a}{z-a}=\frac{xy+xz+yz}{2zz+xz+yz-xy} \text{ seu } xy+xz+yz=a(x+y+z)$$

 quae

quae simplicissima forma statim colligitur ex aequatione

$$\frac{2z(x+y+z)}{xy+xz+yz}=\frac{1+Z}{Z}=\frac{z}{a}.$$

Corollarium.

15. Cum aequationis propositae integrale completum sit

$xy+xz+yz=a(x+y+z)$ seu $\frac{xy+xz+yz}{x+y+z}=\mathrm{Const.}$

ex huius differentiatione etiam ipsa aequatio proposita resultare deprehenditur. Vnde patet aequationem propositam integrabilem reddi si diuidatur per $(x+y+z)^2$ vel etiam per $(xy+xz+yz)^2$.

Scholion.

16. Ex hoc exemplo intelligitur, determinationem functionis per integrationem illatae interdum haud exiguis difficultatibus esse obnoxiam; siquidem hic functionem Z non sine ambagibus elicuimus. Verum et hic ista inuestigatio multo facilius institui potuisset; statim enim atque inuenimus

$$\frac{xy+xz+yz}{2zz+xz+yz-xy}=Z=f:z,$$

hanc ipsam expressionem concinniorem reddere licuisset. Nempe cum sit

$\frac{1}{Z}=\frac{2zz+xz+yz-xy}{xy+xz+yz}$, erit

$1+\frac{1}{Z}=\frac{2z(x+y+z)}{xy+xz+yz}$, ideoque

$\frac{xy+xz+yz}{x+y+z}=\frac{2Zz}{1+Z}=f:z.$

Relicta

Relicta ergo functione Z ſtatim ponatur

$$\frac{xy+xz+yz}{x+y+z}=\Sigma=f:z,$$

et ſumtis differentialibus per ſe liquebit, fieri $d\Sigma=0$, ideoque $\Sigma=$ Conſt. Adhuc facilius hoc problema reſoluitur, ſi etiam ſumto y conſtante eius integrale quaeratur, tum enim ſimili modo peruenitur ad huiusmodi aequationem

$$\frac{xy+xz+yz}{x+y+z}=Y=f:y;$$

quare cum haec expreſſio aeque eſſe debeat functio ipſius z atque ipſius y, neceſſe eſt, vt ea ſit conſtans; eritque propterea aequatio integralis completa

$$xy+xz+yz=a(x+y+z).$$

Exemplum 4.

17. *Huius aequationis differentialis realis:*

$$dx(xx-yy+zz)-zzdy+zdz(y-x)+\frac{xdz}{z}(yy-xx)=0$$

aequationem integralem completam inueſtigare.

Realitas huius aequationis ita oſtenditur.

Ob $P=xx-yy+zz$ erit $L=-3z-\frac{2xy}{z}$

$Q=-zz$ $M=-3z+\frac{yy}{z}-\frac{2xx}{z}$

$R=z(y-x)+\frac{x}{z}(yy-xx)$ $N=-2y$

vnde calculo ſubducto formula $LP+MQ+NR$ euaneſcit.

Sumamus iam z constans, et habebimus hanc aequationem:

$$dx(xx-yy+zz)-zzdy=0,$$

cuius quidem integratio non constaret, nisi perspiceremus ei satisfacere particulariter $y=x$. Hinc autem ponendo $y=x+\frac{zz}{v}$ integrale completum eruere poterimus; fit enim

$$dx\left(zz-\frac{2xzz}{v}-\frac{z^4}{vv}\right)-zzdx+\frac{z^4dv}{vv}=0$$

hincque $dv-\frac{2xvdx}{zz}=dx$,

quae per $e^{\frac{-xx}{zz}}$ multiplicata praebet integrale

$$e^{\frac{-xx}{zz}}v=\int e^{\frac{-xx}{zz}}dx+f:z$$

vbi quidem notandum est in integratione formulae $\int e^{\frac{-xx}{zz}}dx$ quantitatem z vt constantem tractari, esseque $v=\frac{zz}{y-x}$: ita vt sit

$$\int e^{\frac{-xx}{zz}}dx=\frac{e^{\frac{-xx}{zz}}zz}{y-x}+Z.$$

Quodsi iam hanc aequationem differentiare velimus sumta etiam z variabili, difficultas hic occurrit, quomodo quantitatis $\int e^{\frac{-xx}{zz}}dx$ differentiale ex variabilitate ipsius z oriundum definiri debeat. Hic ex principiis repeti debet si fuerit $dV=Sdx+Tdz$, fore

fore $\left(\frac{dT}{dx}\right)=\left(\frac{dS}{dz}\right)$ ideoque si z constans sumatur $T=\int dx\left(\frac{dS}{dz}\right)$. Iam nostro casu est

$$S=e^{\frac{-xx}{zz}} \quad \text{et} \quad V=\int e^{\frac{-xx}{zz}}dx,$$

sumta z constante quare cum sit

$$\left(\frac{dS}{dz}\right)=e^{\frac{-xx}{zz}}\cdot\frac{2xx}{z^3}, \text{ ergo } T=\frac{2}{z^3}\int e^{\frac{-xx}{zz}}xxdx.$$

Quocirca quantitatis $\int e^{\frac{-xx}{zz}}dx$ differentiale plenum ex variabilitate vtriusque x et z oriundum est

$$e^{\frac{-xx}{zz}}dx+\frac{2dz}{z^3}\int e^{\frac{-xx}{zz}}xxdx$$

cui aequari debet alterius partis $\frac{e^{\frac{-xx}{zz}}zz}{y-x}+Z$ differentiale, quod est

$$e^{\frac{-xx}{zz}}\left(\frac{2zdz}{y-x}-\frac{zzdy+zzdx}{(y-x)^2}+\frac{2xxdz-2xzdx}{z(y-x)}\right)+dZ.$$

Turbat vero adhuc formula integralis $\int e^{\frac{-xx}{zz}}xxdx$, in qua z pro constante habetur: reduci autem potest ad priorem $\int e^{\frac{-xx}{zz}}dx$, si ponatur

$$\int e^{\frac{-xx}{zz}}xxdx=Ae^{\frac{-xx}{zz}}x+B\int e^{\frac{-xx}{zz}}dx$$

prodit enim sola x pro variabili habita, differentiando

$$xxdx=Adx-\frac{2Axxdx}{zz}+Bdx \text{ ergo}$$

$$A=-\tfrac{1}{2}zz \text{ et } B=-A=\tfrac{1}{2}zz,$$

 ita

ita vt sit

$$\int e^{\frac{-xx}{zz}} xx\,dx = -\tfrac{1}{2} e^{\frac{-xx}{zz}} xzz + \tfrac{1}{2} zz \int e^{\frac{-xx}{zz}} dx$$

quare cum sit

$$\int e^{\frac{-xx}{zz}} dx = \frac{e^{\frac{-xx}{zz}} zz}{y-x} + Z \text{ erit}$$

$$\int e^{\frac{-xx}{zz}} xx\,dx = -\tfrac{1}{2} e^{\frac{-xx}{zz}} xzz + \frac{e^{\frac{-xx}{zz}} z^4}{2(y-x)} + \tfrac{1}{2} Zzz.$$

Facta ergo substitutione haec orietur aequatio differentialis

$$e^{\frac{-xx}{zz}} \left(dx - \frac{xdz}{z} + \frac{zdz}{y-x}\right) + \frac{Zdz}{z} =$$

$$e^{\frac{-xx}{zz}} \left(\frac{2zdz}{y-x} - \frac{zzdy}{(y-x)^2} + \frac{zzdx}{(y-x)^2} - \frac{2xdx}{y-x} + \frac{2xxdz}{z(y-x)}\right) + dZ$$

quae transit in hanc formam

$$e^{\frac{-xx}{zz}} \left(\frac{dx(y+x)}{y-x} - \frac{zzdx}{(y-x)^2} + \frac{zzdy}{(y-x)^2} - \frac{zdz}{y-x} - \frac{x(y+x)dz}{z(y-x)}\right) = \frac{zdZ - Zdz}{z}$$

seu

$$\frac{e^{\frac{-xx}{zz}}}{(y-x)^2}(dx(yy-xx-zz) + zzdy - zdz(y-x) - \tfrac{xdz}{z}(yy-xx)) = \frac{zdZ-Zdz}{z}$$

qua cum proposita collata euidens est esse debere

$$zdZ - Zdz = 0 \text{ seu } Z = nz;$$

ita vt aequationis propositae integrale completum sit:

$$\int e^{\frac{-xx}{zz}} dx = \frac{e^{\frac{-xx}{zz}} zz}{y-x} + nz,$$

siqui-

siquidem in integrali $\int e^{\frac{-xx}{zz}}dx$ quantitas z pro constante habeatur.

Corollarium.

18. Aequatio ergo proposita integrabilis redditur, si multiplicetur per $\frac{1}{(y-x)^2} e^{\frac{-xx}{zz}}$; ac tum integrale est ipsa aequatio, quam inuenimus.

Scholion 1.

19. Exemplum hoc imprimis est notatu dignum, quod in eius solutione quaedam artificia sunt in subsidium vocata, quibus in praecedentibus non erat opus. Per formulam autem $\int e^{\frac{-xx}{zz}}dx$ integrale non satis determinatum videtur. Cum enim in ea z constans ponatur, constans per integrationem introducenda per nz non definitur, siquidem lex non praescribitur secundum quam integrale $\int e^{\frac{-xx}{zz}}dx$ capi oporteat; vtrum ita vt euanescat facto $x=0$, an alio quocunque modo? Dubium autem hoc diluetur, si aequationem inuentam per z diuidamus, vt formula integralis sit $\int e^{\frac{-xx}{zz}}\frac{dx}{z}$; vbi cum $\frac{dx}{z}$ sit $d.\frac{x}{z}$, euidens est ea exprimi functionem quandam ipsius $\frac{x}{z}$; ac si ponatur $\frac{x}{z}=p$, fore aequationem nostram integralem

$$\int e^{-pp}dp + \text{Const.} = e^{-pp}\frac{z}{y-x}$$

neque

neque hic amplius conditio illa, qua in formula integrali quantitas z pro constante sit habenda, locum habet, sed integrale perinde determinatur, ac si aequatio duas tantum variabiles contineret. Hanc circumstantiam si perpendissemus, plenum differentiale formulae $\int e^{\frac{-xx}{zz}}dx$, ex variabilitate vtriusque x et z nullam difficultatem peperisset. Postquam enim peruenimus ad aequationem

$$\int e^{\frac{-xx}{zz}}dx = e^{\frac{-xx}{zz}} \cdot \frac{zz}{y-z} + f:z$$

eam ita repraesentemus:

$$\int e^{\frac{-xx}{zz}}\frac{dx}{z} = \int e^{\frac{-xx}{zz}} d.\frac{x}{z} = e^{\frac{-xx}{zz}} \cdot \frac{z}{y-z} + Z,$$

vbi cum in formulam integralem etiam variabilitas ipsius z sit inducta, si ea differentietur sumtis omnibus x, y et z variabilibus orietur:

$$e^{\frac{-xx}{zz}}\left(\frac{dx}{z} - \frac{xdz}{zz}\right) = e^{\frac{-xx}{zz}}\left(\frac{dz}{y-z} + \frac{zdx-zdy}{(y-x)^2} - \frac{2xdx}{z(y-x)} + \frac{2xxdz}{zz(y-x)}\right) + dZ$$

seu

$$e^{\frac{-xx}{zz}}\left(\frac{dx(y+x)}{z(y-x)} - \frac{zdx}{(y-x)^2} + \frac{zdy}{(y-x)^2} - \frac{xdz(y+x)}{zz(y-x)} - \frac{dz}{y-x}\right) = dZ$$

quae reducitur ad hanc formam:

$$\frac{e^{\frac{-xx}{zz}}}{z(y-x)^2}\left(dx(yy-xx-zz) + zzdy - zdz(y-x) - \frac{xdz}{z}(yy-xx)\right) = dZ$$

vnde patet esse debere $dZ = 0$ et $Z = \text{Const.}$ sicque elicitur aequatio integralis ante inuenta.

Scho-

Scholion 2.

20. Idem integrale prodiisset, si loco z altera reliquarum x vel y pro constante fuisset assumta; vbi in genere notari conuenit, si huiusmodi aequationem:

$$Pdx+Qdy+Rdz=0$$

sumta z constante tractare licuerit, etiam resolutionem, quaecunque trium variabilium pro constante assumatur, succedere debere, etiamsi id quandoque minus perspiciatur. Ita in aequatione proposita si y pro constante habeatur, resoluenda erit haec aequatio:

$$dx(xx+zz-yy)-zdz(x-y)-\frac{xdz}{z}(xx-yy)=0$$

quae per z multiplicata cum in hanc formam abeat

$$(zdx-xdz)(xx+zz-yy)+yzzdz=0$$

facile patet eam simpliciorem reddi ponendo $x=pz$ tum enim ob

$$zdx-xdz=zzdp$$

prodibit

$$dp(ppzz+zz-yy)+ydz=0$$

fit porro $z=qy$, fietque

$$dp(ppqq+qq-1)+dq=0$$

 cui

cui cum satisfaciat $q=\frac{1}{p}$, statuatur $q=\frac{1}{p}+\frac{1}{r}$ habebiturque

$$dp\left(\frac{2p}{r}+\frac{pp}{rr}+\frac{1}{pp}+\frac{2}{pr}+\frac{1}{rr}\right)-\frac{dp}{pp}-\frac{dr}{rr}=0$$

seu $dp(2ppr+p^3+2r+p)-pdr=0$

vel $dr-\frac{2rdp(pp+1)}{p}=dp(pp+1)$

quae multiplicata per $\frac{1}{pp}e^{-pp}$ et integrata dat

$$e^{-pp}\frac{r}{pp}=\int e^{-pp}\frac{dp(1+pp)}{pp}.$$

At $\int e^{-pp}\frac{dp}{pp}=-e^{-pp}\frac{1}{p}-2\int e^{-pp}dp$,

vnde $e^{-pp}\left(\frac{r}{pp}+\frac{1}{p}\right)=-\int e^{-pp}dp$.

Cum nunc sit

$$p=\frac{x}{z} \text{ et } \frac{1}{r}=\frac{z}{y}-\frac{z}{x}=\frac{z(x-y)}{xy} \text{ erit}$$

$$r=\frac{xy}{z(x-y)},\ \frac{r}{pp}=\frac{yz}{x(x-y)} \text{ et } \frac{r}{pp}+\frac{1}{p}=\frac{z}{x-y}.$$

Vnde aequatio nostra integralis erit

$$\int e^{\frac{-xx}{zz}}d.\frac{x}{z}=e^{\frac{-xx}{zz}}.\frac{z}{y-x}+f:y$$

cuius differentiale, si etiam y pro variabili habeatur, cum aequatione proposita comparatum dabit vt ante $f:y=$ Const.

Ceterum cum in his exemplis variabiles x, y, z vbique eundem dimensionum numerum impleant, methodum generalem huiusmodi aequationes tractandi exponam.

Proble-

Problema 3.

21. Si in aequatione differentiali

$$Pdx + Qdy + Rdz = 0$$

functiones P, Q, R fuerint homogeneae ipsarum x, y et z eiusdem numeri dimensionum; eius integrationem, si quidem fuerit realis, inuestigare.

Solutio.

Sit n numerus dimensionum, quas ternae variabiles x, y et z in functionibus P, Q, R constituunt; ac posito $x = pz$ et $y = qz$, fiet

$$P = z^n S;\ Q = z^n T \text{ et } R = z^n V$$

ita vt iam S, T, V futurae sint functiones binarum tantum variabilium p et q. Cum iam sit

$$dx = pdz + zdp \text{ et } dy = qdz + zdq$$

aequatio nostra hanc induet formam:

$$dz(pS + qT + V) + Szdp + Tzdq = 0$$

seu $\frac{dz}{z} + \frac{Sdp + Tdq}{pS + qT + V} = 0$

quae aequatio realis esse nequit, nisi formula differentialis binas variabiles p et q inuoluens $\frac{Sdp + Tdq}{pS + qT + V}$ per se fuerit integrabilis; quod eueniet si fuerit:

$$(qT + V)\left(\frac{dS}{dq}\right) + pT\left(\frac{dS}{dp}\right) - (pS + V)\left(\frac{dT}{dp}\right) - qS\left(\frac{dT}{dq}\right)$$
$$- S\left(\frac{dV}{dq}\right) + T\left(\frac{dV}{dp}\right) = 0.$$

Quoties ergo hic character locum habet, nostra aequatio erit realis; eiusque integrale erit

$$lz + \int \frac{Sdp + Tdq}{pS + qT + V} = \text{Const.}$$

vbi tantum opus est vt loco litterarum p et q valores assumti $\frac{x}{z}$ et $\frac{y}{z}$ restituantur.

Coroll. 1

22. Ita in nostro primo exemplo (§. 12.) cum sit

$$P = y + z;\ Q = x + z;\ R = x + y \text{ erit}$$

$$S = q + 1;\ T = p + 1;\ V = p + q \text{ et}$$

$$\frac{dz}{z} + \frac{(q+1)dp + (p+1)dq}{2pq + 2p + 2q} = 0;$$

cuius integrale est

$$lz + \tfrac{1}{2}l(pq + p + q) = \tfrac{1}{2}l(xy + xz + yz) = C$$

seu $xy + xz + yz = C.$

Coroll. 2.

23. In secundo exemplo (§. 13.) est

$$P = ay - bz;\ Q = cz - ax;\ R = bx - cy \text{ hinc}$$

$$S = aq - b;\ T = c - ap;\ V = bp - cq.$$

Ergo $\frac{dz}{z} + \frac{(aq-b)dp + (c-ap)dq}{0} = 0$

hincque

$$(aq - b)dp + (c - ap)dq = 0$$

et integrando

$$l\frac{aq - b}{c - ap} = l\frac{ay - bz}{cz - ax} = C.$$

Coroll. 3.

Coroll. 3.

24. In tertio exemplo (§. 14.) fit

$S = qq + q + 1$; $T = pp + p + 1$; et $V = pp + pq + qq$

hincque

$$\frac{dz}{z} + \frac{dp(qq+q+1) + dq(pp+p+1)}{ppq + pqq + pp + 3pq + qq + p + q} = 0$$

qui denominator est $= (p+q+1)(pq+p+q)$, vnde haec fractio resoluitur in has duas

$$\frac{-dp-dq}{p+q+1} + \frac{dp(q+1) + dq(p+1)}{pq+p+q}$$

ex quo integrale a logarithmis ad numeros perductum oritur

$$\frac{z(pq+p+q)}{p+q+1} = \frac{xy+xz+yz}{x+y+z} = C.$$

Coroll. 4.

25. In exemplo quarto (§. 17.) fit

$S = pp - qq + 1$; $T = -1$; $V = q - p + p(qq - pp)$,

hincque

$$\frac{dz}{z} + \frac{dp(pp-qq+1) - dq}{0} = 0$$

ideoque

$$dq = dp(pp - qq + 1).$$

Cum ergo satisfaciat $q = p$ ponatur $q = p + \frac{1}{r}$, fiet $dr - 2pr\,dp = dp$; et integrando:

$$e^{-pp} r = \int e^{-pp} dp = e^{-pp} \cdot \frac{1}{q-p};$$

ita vt integrale sit

$$e^{\frac{-xx}{zz}} \cdot \frac{z}{y-x} = \int e^{\frac{-xx}{zz}} d.\frac{x}{z} + \text{Const.}$$

Scholion.

26. Cum igitur aequationes differentiales tres variabiles inuoluentes nullam habeant difficultatem sibi propriam, quoniam earum resolutio, siquidem fuerint reales, semper ad aequationes differentiales duarum variabilium reduci potest; hoc argumentum fusius non prosequor. Quod enim ad eiusmodi aequationes differentiales trium variabilium attinet, in quibus ipsa differentialia ad plures dimensiones ascendunt, veluti est

$$Pdx^2 + Qdy^2 + Rdz^2 + 2Sdxdy + 2Tdxdz + 2Vdydz = 0$$

de iis generatim tenendum est, nisi per radicis extractionem ad formam

$$Pdx + Qdy + Rdz = 0$$

reduci queant, eas semper esse absurdas. Quomodocunque enim aequatio integralis esset comparata, ex ea valor ipsius z ita definiri posset, vt z aequetur functioni binarum variabilium x et y, vnde foret $dz = pdx + qdy$; neque hae variabiles x et y vllo modo a se penderent. Hic ergo valor $pdx + qdy$ loco dz in aequatione differentiali substitutus, ita satisfacere deberet, vt omnes termini se mutuo destruerent, quod autem fieri non posset, si ex aequaquationis resolutione dz ita definiretur, vt differentialia dx et dy signis radicalibus essent inuoluta. Hinc aequatio illa exempli loco allata, cum per resolutionem det:

$$dz = \frac{-Tdx - Vdy \pm \surd((TT-PR)dx^2 + 2(TV-RS)dxdy + (VV-QR)dy^2)}{R}$$

realis esse nequit, nisi radix extrahi queat, hoc est nisi ipsa aequatio in factores formae

$$Pdx + Qdy + Rdz$$

resolui possit. Atque etiamsi hoc eueniat, et hi factores nihilo aequales statuantur, tamen aequatio non erit realis, nisi criterium supra traditum locum habeat. Ex his perspicuum est, ne eiusmodi quidem aequationes, quae quatuor pluresue variabiles inuoluant, plus difficultatis habere.

Problema 4.

27. Si V sit functio quaecunque binarum variabilium x et y, in formula autem integrali $\int V dx$ quantitas y pro constante sit habita, definire huius formae $\int V dx$ differentiale, si praeter x etiam y variabilis assumatur.

Solutio.

Ponatur ista formula integralis $\int V dx = Z$, eritque Z vtique functio ambarum variabilium x et y, etiamsi in ipsa integratione y pro constante habeatur. Euidens autem est, si vicissim in differentiatione y constans sumatur, fore $dZ = V dx$. Quare si etiam y varia-

variabilis statuatur, differentiale ipsius $Z=\int V dx$ huiusmodi habebit formam:

$$dZ = V dx + Q dy$$

et quaestio huc redit, vt ista quantitas Q determinetur. Quia autem forma $V dx + Q dy$ est verum differentiale, necesse est sit $(\frac{dV}{dy}) = (\frac{dQ}{dx})$; hincque $dx(\frac{dQ}{dx}) = dx(\frac{dV}{dy})$: At $dx(\frac{dQ}{dx})$ est differentiale ipsius Q, si y pro constante habeatur; vnde Q reperietur si formula $dx(\frac{dV}{dy})$ ita integretur, vt y tanquam constans tractetur, seu erit $Q=\int dx(\frac{dV}{dy})$. Quocirca formulae $Z=\int V dx$ differentiale ex variabilitate vtriusque x et y oriundum erit

$$dZ = V dx + dy.\int dx(\frac{dV}{dy}).$$

Coroll. 1.

28. Quoniam V est functio ipsarum x et y, si ponatur $dV = R dx + S dy$, erit $S = (\frac{dV}{dy})$; vnde fit

$$dZ = d.\int V dx = V dx + dy \int S dx$$

scilicet in formulae $\int S dx$ integratione, perinde ac formulae $\int V dx$ sola quantitas x pro variabili est habenda.

Coroll. 2.

29. Si V fuerit functio homogenea ipsarum x et y existente numero dimensionum $=n$, posito $dV = R dx + S dy$ erit $Rx + Sy = nV$, ideoque $S = \frac{nV}{y} - \frac{Rx}{y}$ hinc

hinc $\int S\,dx = \frac{n}{y}\int V\,dx - \frac{1}{y}\int Rx\,dx$. At ob y conſtans eſt $R\,dx = dV$ hinc $\int R\,x\,dx = \int x\,dV = Vx - \int V\,dx$, ideoque $\int S\,dx = \frac{n+1}{y}\int V\,dx - \frac{Vx}{y}$, et

$$dZ = d.\int V\,dx = V\,dx - \frac{Vx\,dy}{y} + \frac{(n+1)\,dy}{y}\int V\,dx.$$

Coroll. 3.

30. Idem facilius inuenitur ex conſideratione quod functio $Z = \int V\,dx$ futura ſit homogenea $n+1$ dimenſionum, quare poſito $dZ = V\,dx + Q\,dy$, erit $Vx + Qy = (n+1)Z$; ideoque $Q = \frac{(n+1)Z}{y} - \frac{Vx}{y}$, vt ante.

Scholion.

31. Problemate iam ante, et in praecedente quidem libro ſum vſus, neque tamen abs re fore putaui, ſi id data opera hic tractarem, quandoquidem hic liber in functionibus binarum pluriumue variabilium occupatur. Praecipuum autem negotium non in eiusmodi aequationibus differentialibus, quales in hoc capite integrare docui, verſatur, quod quidem breui eſſet abſolutum, ſed cum differentiatio functionis binarum variabilium x et y duplices formulas $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$ ſuppeditet, exiſtente V huiusmodi functione, hoc loco eiusmodi quaeſtiones potiſſimum contemplabimur, quibus talis functio V ex data quacunque relatione harum duarum formularum $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$ eſt definienda. Relatio autem haec per aequationem inter iſtas formulas et binas variabiles x et y, quam etiam ipſa functio quaeſita V ingredi poteſt, exprimitur, ex cuius aequationis in-

dole diuisio tractationis erit petenda. Problema scilicet generale, in quo soluendo ista sectio est occupata, ita se habet, vt ea binarum variabilium x et y functio V inueniatur, quae satisfaciat aequationi cuicunque inter quantitates x, y, V, $(\frac{dV}{dx})$ et $(\frac{dV}{dy})$ propositae. Quodsi in hanc aequationem altera tantum binarum formularum differentialium $(\frac{dV}{dx})$ vel $(\frac{dV}{dy})$ ingrediatur, resolutio non est difficilis, atque ad casum aequationum differentialium duas tantum variabiles inuoluentium reducitur; quando autem ambae istae formulae in aequatione proposita insunt, quaestio multo magis est ardua ac saepe numero ne resolui quidem potest, etiamsi resolutio aequationum differentialium duas tantum variabiles complectentium admittatur: in hoc enim negotio, quoties resolutionem ad integrationem aequationum differentialium inter duas variabiles reducere licet, problema pro resoluto erit habendum. Cum igitur ex aequatione proposita formula $(\frac{dV}{dy})$ aequetur functioni vtcunque ex quantitatibus x, y, V et $(\frac{dV}{dx})$ conflatae, ex indole huius functionis, prout fuerit simplicior, et vel solam formulam $(\frac{dV}{dx})$ vel praeter eam vnicam ex reliquis, vel etiam binas vel adeo omnes comprehendat, tractationem sequentem distribuemus. Hoc enim ordine seruato facillime apparebit, quantum adhuc praestare liceat, et quantum adhuc desideretur. Praeterea vero nonnulla subsidia circa transformationem binarum formularum differentialium ad alias variabiles exponenda occurrent.

Diuisio

Diuiſio huius Sectionis.

32. Quo partes, quas in hac ſectione pertractari conuenit, clarius conſpectui exponantur, quoniam hae quaeſtiones circa functiones binarum variabilium verſantur, ſint x et y binae variabiles, et z earum functio ex data quadam differentialium relatione definienda, ita vt aequatio finita inter x, y et z requiratur. Ponamus autem $dz = pdx + qdy$, ita vt ſit recepto ſignandi modo $p = (\frac{dz}{dx})$ et $q = (\frac{dz}{dy})$, atque ideo p et q ſint formulae differentiales, quae in relationem propoſitam ingrediantur. In genere ergo relatio iſta erit aequatio quaecunque inter quantitates p, q, x, y et z propoſita, atque haec ſectio perfecte abſolueretur, ſi methodus conſtaret, ex data aequatione quacunque inter has quantitates p, q, x, y et z eruendi aequationem inter x, y et z; quod autem cum in genere ne pro functionibus quidem vnicae variabilis praeſtari poſſit, multo minus hic eſt expectandum, ex quo eos caſus tantum euolui conueniet, qui reſolutionem admittant. Primo autem reſolutio ſuccedit, ſi in aequatione propoſita altera formularum differentialium p vel q plane deſit, ita vt aequatio vel inter p, x, y et z vel inter q, x, y et z proponatur. Deinde aequationes, quae ſolas binas formulas differentiales p et q continent, ita vt altera debeat eſſe functio quaecunque alterius, commode reſoluere licet. Tum igitur ſequentur aequationes, quae praeter p et q vnicam quantitatum finitarum x vel y vel z complectantur, ex quo genere

cuiusmodi casus resolui queant videamus. Ordo porro postulat, vt ad aequationes, quae praeter binas formulas differentiales p et q insuper binas quantitatum finitarum vel x et y, vel x et z, vel y et z, inuoluunt, progrediamur; ac denique de resolutione aequationum omnes litteras p, q, x, y et z implicantium, agemus, subsidia transformationis deinceps exposituri.

CAPVT II.

CAPVT II.

DE RESOLVTIONE AEQVATIONVM QVIBVS ALTERA FORMVLA DIFFERENTIALIS PER QVANTITATES FINITAS VTCVNQVE DATVR.

Problema 4.

33.

Inuestigare indolem functionis z binarum variabilium x et y, vt formula differentialis $(\frac{dz}{dx}) = p$ sit quantitas constans $= a$.

Solutio.

Posito ergo $dz = pdx + qdy$, ea functionis z indoles quaeritur vt sit $p = a$, seu $dz = adx + qdy$; ad quam inueniendam sumatur y pro constante, erit $dz = adx$, et integrando $z = ax +$ Const. vbi notari oportet hanc constantem vtcunque inuoluere posse quantitatem y. Quare vt solutionem generalem exhibeamus erit $z = ax + f:y$, denotante $f:y$ functionem quamcunque ipsius y, quae per se nullo modo determinatur, sed penitus ab arbitrio nostro pendet

Quod etiam differentiatio vicissim declarat, si enim huius functionis $f:y$ differentiale per $dyf':y$ indicemus, erit vtique $dz = adx + dyf':y$; ideoque $\left(\frac{dz}{dx}\right) = a$, prorsus vti quaestio postulat; vnde patet hoc casu alteram formulam differentialem $q = \left(\frac{dz}{dy}\right)$, functioni solius y aequari, cum sit $q = \left(\frac{dz}{dy}\right)$.

Coroll. 1.

34. Si ergo eiusmodi quaeratur functio z binarum variabilium x et y, vt sit $\left(\frac{dz}{dx}\right) = a$, erit $z = ax + f:y$, et altera formula differentialis $\left(\frac{dz}{dy}\right)$ necessario aequatur functioni ipsius y tantum.

Coroll. 2.

35. Si talis requiratur functio, vt sit $\left(\frac{dz}{dx}\right) = 0$, ea necessario erit functio ipsius y tantum, seu quantitatem x plane non inuoluet; cum enim a variatione ipsius x nullam mutationem pati debeat, haec quantitas x quoque in eius determinationem plane non ingredietur.

Coroll. 3.

36. Hinc etiam patet aequationem differentialem $dz = adx + qdy$ realem esse non posse, nisi q sit functio ipsius y tantum; quod etiam character supra expositus declarat, aequatione enim ad hanc formam $adx + qdy - dz = 0$ reducta, ob $P = a$, $Q = q$ et $R = -1$ erit $L = \left(\frac{dq}{dz}\right)$; $M = 0$, et $N = -\left(\frac{dq}{dx}\right)$; ideo-

ideoque realitas postulat vt sit $a(\frac{dq}{dz})+(\frac{dq}{dx})=0$. At per hypothesin q non pendet a z, vnde ob $(\frac{dq}{dz})=0$; erit $(\frac{dq}{dx})=0$, ideoque etiam q ab x non pendet.

Scholion 1.

37. Ex allatis satis patet hanc operationem, qua functionem z determinauimus veram esse integrationem, qua vti in vulgaribus integrationibus aliquid indeterminati introducitur. Hic scilicet ingressa est functio quaecunque ipsius y, cuius indoles per se nullo modo determinatur; eam quoque ita concipere licet, vt descripta curua quacunque, si eius abscissae per y indicentur, applicatae exhibeant eiusmodi functionem ipsius y. Neque vero opus est, vt haec curua sit regularis et aequatione quapiam contenta; sed curua quaecunque libero manus ductu descripta eundem praestat effectum, etiamsi sit maxime irregularis, et ex pluribus partibus diuersarum curuarum conflata. Huiusmodi functiones irregulares appellare licet discontinuas seu nexu continuitatis destitutas; vnde hoc imprimis notatu dignum occurrit, quod cum prioris generis integrationes alias functiones praeter continuas non admittant, hic etiam functiones discontinuae calculo subiiciantur, quod pluribus insignibus Geometris adeo calculi principiis aduersari est visum. Verum integrationum in hoc secundo libro tradendarum vis praecipua in eo consistit, quod etiam functionum discontinuarum sint capaces

paces; ex quo per hunc quasi nouum calculum fines Analyseos maxime proferri sunt censendi.

Scholion 2.

38. Quemadmodum deinde in vulgaribus integrationibus constans arbitraria ingressa, semper ex indole problematis, cuius solutio eo perduxerat, determinatur, ita etiam hic natura problematis, cuius solutio huiusmodi integratione absoluitur, semper indolem functionis arbitrariae per integrationem ingressae determinabit. Ita si cordae tensae figura quaecunque inducatur, eaque subito dimittatur, vt oscillationes peragat, ope principiorum mechanicorum ad quodvis tempus figura, quam corda tum sit habitura, definiri potest, hocque fit eiusmodi integratione, qua functio quaedam arbitraria introducitur; quam autem deinceps ita determinari conuenit, vt pro ipso motus initio ipsa illa figura cordae inducta prodeat; et cum solutio debeat esse generalis, vt satisfaciat figurae cuicunque initiali, necesse est vt etiam ad eos casus pateat, quibus cordae initio figura irregularis nullo continuitatis nexu praedita inducatur, quod fieri non posset, nisi per integrationem eiusmodi functio arbitrio nostro relicta ingrederetur, quam etiam ad figuras irregulares adaptare liceret. Huiusmodi functiones arbitrarias, prouti hic feci, eiusmodi signandi modo $f:y$ indicabo, vnde cauendum erit ne littera f pro quantitate habeatur, quocirca ipsi *colon* suffigere visum est. Simili modo

modo in sequentibus haec scriptio $f:(x+y)$ denotabit functionem arbitrariam quantitatis $x+y$; ac vbi plures tales functiones in calculum ingredientur, praeter litteram f etiam his characteribus ϕ, ψ, θ etc. cum simili significatione vtar.

Problema 5.

39. Inuestigare indolem functionis z binarum variabilium x et y, vt formula differentialis $(\frac{dz}{dx})=p$ aequalis fiat functioni datae ipsius x, quae sit X, ita vt sit $p=X$.

Solutio.

Posito $dz=pdx+qdy$ ob $p=X$ erit $dz=Xdx+qdy$; quia iam huius differentialis pars Xdx est data, ad integrale inueniendum accipiatur y constans, et cum sit $dz=Xdx$, erit integrando $z=\int Xdx+$Const. quae constans cum etiam quantitatem y vtcunque implicare possit, pro ea assumere licebit functionem quamcunque arbitrariam ipsius y, eritque ergo integrale quaesitum $z=\int Xdx+f:y$, quae per differentiationem praebet $dz=Xdx+dyf':y$, ita vt sit $q=f':y$, atque $(\frac{dz}{dx})=X$ plane, vt requirebatur.

Coroll. 1.

40. Aequationis ergo $(\frac{dz}{dx})=X$, existente z functione duarum variabilium x et y, integrale est $z=\int Xdx+f:y$ vbi ob X datum, formula integra-

lis $\int X dx$ datam functionem ipsius x denotat; quandoquidem constans hac integratione ingressa in functione arbitraria $f:y$ comprehendi potest.

Coroll. 2.

41. Hinc sequitur aequationem differentialem $dz = X dx + q dy$ realem esse non posse, nisi q sit functio ipsius y; quod quidem cum hac limitatione est intelligendum, nisi q etiam inuoluat quantitatem z; quem casum autem hinc remouemus.

Scholion.

42. Si enim q etiam a z pendere queat, aequatio $dz = X dx + q dy$ realis erit, si q fuerit functio quaecunque binarum quantitatum $z - \int X dx$ et y; id quod hinc facillime patet si ponatur $z - \int X dx = u$, ita vt iam q futura sit functio binarum quantitatum u et y. Tum enim aequatio differentialis, quae fit $du = q dy$, duas tantum continet variabiles u et y, ideoque certo est realis; et quomodocunque eius integrale se habeat, inde semper u aequabitur certae functioni ipsius y, vnde fit $u = z - \int X dx = f:y$, prorsus vt ante. Quoties ergo esse debet $\left(\frac{dz}{dx}\right) = X$, etiam ne hocquidem casu excepto, quo forte q ipsam quantitatem z implicat, integrale erit

$$z = \int X dx + f:y,$$

neque vnquam alia solutio locum habere potest. Erit ergo hoc integrale completum, propterea quod fun-

functionem arbitrariam inuoluit, id quod pro certissimo criterio integralis completi est habendum. Hic igitur ad integrale completum requiritur, vt non tam constans quaedam arbitraria, sed functio adeo variabilis arbitraria ingrediatur; ita si quis pro casu $\left(\frac{dz}{dx}\right) = axx$ exhibeat hoc integrale

$$z = \tfrac{1}{3}ax^3 + A + By + Cy^2 + \text{etc.}$$

id tantum erit particulare, etiamsi plures constantes arbitrarias A, B, C etc. ac fortasse infinitas complectatur; verum enim integrale completum

$$z = \tfrac{1}{3}ax^3 + f:y$$

infinite latius patet; id quod ad sequentia recte intelligenda probe notari oportet. Occurrent autem vtique casus, quibus ob defectum methodi integrale completum inuestigandi, integralibus particularibus contenti esse debemus, quae etiamsi adeo infinitas constantes arbitrarias comprehendant, tamen pro solutionibus particularibus tantum sunt habenda. Hanc obseruationem in sequentibus perpetuo meminisse oportet, ne circa integralia particularia et completa vnquam decipiamur.

Problema 6.

43. Si z debeat esse eiusmodi functio binarum variabilium x et y, vt formula differentialis $\left(\frac{dz}{dx}\right) = p$ aequetur functioni cuipiam datae ipsarum x et y, definire in genere indolem functionis quaesitae z.

Solutio.

Sit V functio ista data ipsarum x et y, cui formula differentialis $\left(\frac{dz}{dx}\right) = p$ aequalis esse debet, ac posito $dz = p\,dx + q\,dy$ requiritur vt sit $p = V$. Iam ad formam functionis z inueniendam consideretur quantitas y tanquam constans, eritque $dz = V\,dx$. Integretur igitur formula $\int V\,dx$ spectata sola x vt variabili, quia y pro constante sumitur, ita vt in hac formula vnica insit variabilis x, ideoque eius integratio nulli obnoxia sit difficultati; id tantum est tenendum, constantem integratione ingressam vtcunque inuoluere posse alteram quantitatem y, sicque pro functione quaesita z haec habebitur expressio:

$$z = \int V\,dx + f:y$$

integrali $\int V\,dx$ ita sumto, quasi quantitas y esset constans solaque x variabilis; at $f:y$ denotat functionem quamcunque arbitrariam ipsius y ne exclusis quidem formis discontinuis, quae nullis expressionibus analyticis exhiberi queant, atque ob hanc ipsam functionem arbitrariam integratio pro completa est habenda.

Coroll. 1.

44. Cum V sit functio data ipsarum x et y, formula integralis $\int V\,dx$ erit etiam functio cognita et determinata earundem quantitatum x et y, quod enim per integrationem arbitrarii ingreditur, in altera parte $f:y$ comprehenditur.

Coroll. 2.

Coroll. 2.

45. Hinc etiam differentialis dz altera pars $q\,dy$, ex variabilitate ipsius y oriunda definitur. Nam per (27.) est formae $\int V\,dx$ differentiale ex vtraque variabili x et y ortum:

$$V\,dx + dy \int dx \left(\frac{dV}{dy}\right)$$

ac si functionis $f:y$ differentiale indicetur per $dy f':y$ erit

$$dz = V\,dx + dy \int dx \left(\frac{dV}{dy}\right) + dy f':y.$$

Coroll. 3.

46. Cum ergo posuerimus $dz = p\,dx + q\,dy$, sitque $p = V$ erit

$$q = \int dx \left(\frac{dV}{dy}\right) + f':y,$$

vbi ob V functionem datam ipsarum x et y, etiam $\left(\frac{dV}{dy}\right)$ erit functio data, et in integratione $\int dx \left(\frac{dV}{dy}\right)$ sola x pro variabili habetur.

Exemplum 1.

47. *Quaeratur eiusmodi functio* z *ipsarum* x *et* y, *vt sit* $\left(\frac{dz}{dx}\right) = \frac{x}{\sqrt{(xx+yy)}}$.

Ob $V = \frac{x}{\sqrt{(xx+yy)}}$ erit $\int V\,dx = \sqrt{(xx+yy)}$, ideoque habemus

$$z = \sqrt{(xx+yy)} + f:y$$

 vnde

vnde fit

$$\left(\frac{dz}{dy}\right) = q = \frac{y}{\sqrt{(xx+yy)}} + f':y\,,$$

id quod etiam per regulam datam prodit. Erit enim

$$\left(\frac{dV}{dy}\right) = \frac{-xy}{(xx+yy)^{\frac{3}{2}}}$$

hinc sumta y constante

$$\int dx\left(\frac{dV}{dy}\right) = -y\int\frac{x\,dx}{(xx+yy)^{\frac{3}{2}}} = \frac{y}{\sqrt{(xx+yy)}}.$$

Exemplum 2.

48 *Quaeratur eiusmodi functio* z *ipsarum* x *et* y, *vt sit* $\left(\frac{dz}{dx}\right) = \frac{y}{\sqrt{(yy-xx)}}$.

Cum sit $V = \frac{y}{\sqrt{(yy-xx)}}$ erit

$$\int V\,dx = y \text{ Ang. sin.} \frac{x}{y}\,,$$

hincque

$$z = y \text{ Ang. sin.} \frac{x}{y} + f:y$$

cuius differentiale ex ipsius y variabilitate oriundum, si desideremus, ob

$$\left(\frac{dV}{dy}\right) = \frac{-xx}{(yy-xx)^{\frac{3}{2}}} \text{ erit}$$

$$\int dx\left(\frac{dV}{dy}\right) = -\int\frac{xx\,dx}{(yy-xx)^{\frac{3}{2}}} = \int\frac{dx}{\sqrt{(yy-xx)}} - yy\int\frac{dx}{(yy-xx)^{\frac{3}{2}}}$$

ideo-

ideoque

$$\int dx\left(\frac{dV}{dy}\right) = \text{Ang. fin.}\frac{z}{y} - \frac{z}{\sqrt{(yy-xx)}},\ \text{et}$$

$$q = \text{Ang. fin.}\frac{z}{y} - \frac{z}{\sqrt{(yy-xx)}} + f':y.$$

Idem reperitur ex differentiatione expressionis pro z inuentae:

$$dz = dy\,\text{Ang. fin.}\frac{x}{y} + \frac{ydx-xdy}{\sqrt{(yy-xx)}} + dyf':y$$

vnde pro $q = \left(\frac{dz}{dy}\right)$ idem valor prodit.

Exemplum 3.

49. *Quaeratur eiusmodi functio* z *ipsarum* x *et* y, *vt fit* $\left(\frac{dz}{dx}\right) = \frac{a}{\sqrt{(aa-yy-xx)}}$.

Ob $V = \frac{a}{\sqrt{(aa-yy-xx)}}$ erit

$$\int V dx = a\,\text{Ang. fin.}\frac{x}{\sqrt{(aa-yy)}}$$

vnde functionis z forma quaesita est

$$z = a\,\text{Ang. fin.}\frac{x}{\sqrt{(aa-yy)}} + f:y.$$

Deinde quia

$$\left(\frac{dV}{dy}\right) = \frac{ay}{(aa-yy-xx)^{\frac{3}{2}}},\ \text{erit}$$

$$\int dx\left(\frac{dV}{dy}\right) = ay\int\frac{dx}{(aa-yy-xx)^{\frac{3}{2}}} = \frac{ay}{aa-yy}\cdot\frac{x}{\sqrt{(aa-yy-xx)}}.$$

ideoque

$$\left(\frac{dz}{dy}\right) = q = \frac{axy}{(aa-yy)\sqrt{(aa-yy-xx)}} + f':y$$

quae

quae eadem expressio etiam ex ipsa differentiatione ipsius z eruitur.

Scholion 1.

50. In hoc calculo tamen adhuc quaedam incertitudo relinquitur, qua valor quantitatis q afficitur. Cum enim valor ipsius $z = \int V dx + f : y$ sit determinatus, quandoquidem integrale $\int V dx$ respectu ipsius x ita fuerit determinatum, vt pro dato ipsius x valore etiam datum valorem obtineat; adeoque in eius differentiali pleno nulla incertitudo inesse potest, sed necesse est, vt valor ipsius q aeque prodeat determinatus atque ipsius p, interim tamen formula integralis $\int dx (\frac{dV}{dy})$ non determinatur, sed nouam arbitrariam a priori non pendentem introducere videtur. Vt igitur talis significatus vagus euitetur, spectari oportet conditionem, qua integrale $\int V dx$ determinatur, eademque conditio in formulae $\int dx (\frac{dV}{dy})$ integratione adhiberi debet. Nam ponamus integrale $\int V dx$ ita capi vt euanescat posito $x = a$, sitque eius valor determinatus $\int V dx = S$, isque igitur potentia saltem habebit factorem $a - x$ seu $a^n - x^n$; qui cum non contineat y, etiam $(\frac{dS}{dy})$ eundem factorem continebit ideoque $(\frac{dS}{dy})$ euanescet posito $x = a$. Est vero $(\frac{dS}{dy}) = \int dx (\frac{dV}{dy})$, ex quo perspicitur, si integrale $\int V dx$ ita capiatur vt euanescat posito $x = a$, etiam alterum integrale $\int dx (\frac{dV}{dy})$ ita capi debere, vt euanescat posito $x = a$. In allatis binis postremis exemplis, vtraque integratio ita est instituta, vt

euane-

euanefcat pofito $x=0$, in primo autem nulla huiusmodi regula eft obferuata; fin autem eandem legem adhibeamus, habebimus

$$\int V dx = \sqrt{(xx+yy)}-y \text{ et } \int dx\left(\frac{dV}{dy}\right)=\frac{y}{\sqrt{(xx+yy)}}-1,$$

vnde quidem eadem folutio emergit; quia ibi $-y$ continetur in $f:y$ et hic -1 in $f':y$. Perinde autem eft quacunque lege prior integratio determinetur, dummodo eadem lege et in pofteriori vtamur.

Scholion 2.

51. Principium huius determinationis ifto innititur Theoremate aeque elegante ac notatu digno:

Si S *fit eiusmodi functio binarum variabilium* x *et* y *quae euanefcat pofito* $x=a$, *fueritque*

$$dS = Pdx + Qdy,$$

tum etiam quantitas Q *euanefcet pofito* $x=a$.

Vnde fimul colligitur, fi S euanefcat pofito $y=b$, tum etiam fieri $P=0$ fi ponatur $y=b$. Hic autem probe obferuandum eft, quae de fimili determinatione binarum formularum integralium $\int Vdx$ et $\int dx\left(\frac{dV}{dy}\right)$ funt praecepta, tantum valere fi valor a ipfi x tribuendus fuerit conftans; neque etiam fuperius Theorema locum habet, fi verbi gratia functio S euanefcat pofito $x=y$, inde enim neutiquam fequitur eodem cafu quantitatem Q effe euanituram. Etiamfi enim functio S factorem habeat $x-y$ vel

 x^n-y^n,

$x^n - y^n$, minime sequitur formulam $\left(\frac{dS}{dy}\right)$ seu Q eundem factorem esse habituram, quemadmodum vsu venit, si factor fuerit $x-a$ seu $x^n - a^n$. Dixi autem non opus esse, vt talis factor reuera adsit, dummodo quasi potentia in functione S contineatur. Veluti si fuerit

$$S = a - x + y - \sqrt{(aa - xx + yy)},$$

quae functio posito $x = a$ vtique euanescit, etiamsi neque factorem $x-a$ neque $x^n - a^n$ contineat; simul vero etiam $\left(\frac{dS}{dy}\right) = 1 - \frac{y}{\sqrt{(aa-xx+yy)}}$ posito $x = a$ euanescit. In huiusmodi ergo calculo, quo in his problematibus vtimur, vbi integrale formulae $\int V dx$ exhiberi debet, id semper ex duabus partibus compositum spectamus, altera indeterminata per functionem $f:y$ indicata, altera autem, quam proprie per $\int V dx$ exprimimus determinata, quae scilicet posito $x = a$ euanescat; hicque semper perinde est qualis constans pro a assumatur, dum discrimen perpetuo alteri parti indeterminatae inuoluitur.

Problema 7.

52. Si z debeat ita determinari per binas variabiles x et y vt formula differentialis $\left(\frac{dz}{dx}\right) = p$ aequetur datae cuipiam functioni ipsarum y et z, quae sit V, definire in genere indolem functionis z per x et y.

Solutio.

Solutio.

Cum posito $dz = pdx + qdy$, sit $p = V$, si quantitatem y pro constante capiamus, erit $dz = Vdx$, vbi cum V sit functio data ipsarum y et z, et y pro constante habeatur aequatio $\frac{dz}{V} = dx$; erit integrabilis, ex cuius integratione completa oritur

$$\int \frac{dz}{V} = x + f:y,$$

qua aequatione relatio inter ternas variabiles x, y et z ita in genere exprimitur, vt ex ea z per x et y definiri, indolesque functionis z assignari possit.

Quodsi hinc alteram quoque differentialis partem qdy seu functionem $q = (\frac{dz}{dy})$ indagare velimus, ponamus integrale $\int \frac{dz}{V}$, vbi y vt constans spectatur, ita capi vt euanescat posito $z = c$, eritque quantitatem $\int \frac{dz}{V}$ denuo differentiando vt etiam y variabilis assumatur:

$$d.\int \frac{dz}{V} = \frac{dz}{V} + dy \int dz \left(\frac{d(1:V)}{dy}\right) \text{ seu}$$

$$d.\int \frac{dz}{V} = \frac{dz}{V} - dy \int \frac{dz}{VV} \left(\frac{dV}{dy}\right)$$

vbi in integrali $\int \frac{dz}{VV} \left(\frac{dV}{dy}\right)$ quantitas y iterum pro constante habetur, hocque integrale ita capi debet, vt posito $z = c$ euanescat. Quo facto cum aequationis inuentae differentiale sit:

$$\frac{dz}{V} - dy \int \frac{dz}{VV} \left(\frac{dV}{dy}\right) = dx + dy f':y,$$

pro forma propofita habebimus:

$$dz = V\,dx + dy\left(V\int\frac{dz}{VV}\left(\frac{dV}{dy}\right) + Vf':y\right)$$

vnde quantitas q innotefcit.

Coroll. 1.

53. In hoc problemate facillime definitur, qualis functio quantitas x futura fit binarum reliquarum y et z, cum fit

$$x = \int\frac{dz}{V} - f:y;$$

fiquidem V per y et z detur. Perinde autem eft fiue z per x et y, fiue x per y et z determinetur.

Coroll. 2.

54. Cum relatio inter ternas variabiles x, y et z ita fit determinata, vt fiat $\left(\frac{dz}{dx}\right) = V$ functioni datae ipfarum y et z, ob $dx = \frac{dz}{V}$, fumto y conftante, erit x eiusmodi functio ipfarum y et z vt fit $\left(\frac{dx}{dz}\right) = \frac{1}{V}$, ideoque $\left(\frac{dz}{dx}\right).\left(\frac{dx}{dz}\right) = 1$.

Scholion.

55. In genere autem quaecunque relatio inter ternas variabiles x, y et z proponatur, vnde vnaquaecque per binas reliquas determinari et tanquam earundem functio fpectari poffit; femper erit $\left(\frac{dz}{dx}\right).\left(\frac{dx}{dz}\right) = 1$. Ponamus enim aequatione illam relationem exprimente differentiata prodire

$$P\,dx + Q\,dy + R\,dz = 0;$$

ac

ac manifestum est sumta y pro constante fore

$$Pdx + Rdz = 0,$$

ideoque tam $\left(\frac{dz}{dx}\right) = \frac{-P}{R}$ quam $\left(\frac{dx}{dz}\right) = \frac{-R}{P}$; simili autem modo erit:

$$\left(\frac{dx}{dy}\right) = \frac{-Q}{P};\ \left(\frac{dy}{dx}\right) = \frac{-P}{Q};\ \left(\frac{dz}{dy}\right) = \frac{-Q}{R};\ \left(\frac{dy}{dz}\right) = \frac{-R}{Q}$$

vnde propositum patet, etiamsi relatio inter plures tribus variabiles locum habeat Ceterum hic casus a praecedentibus differt, quod hic natura functionis z, quatenus ex binis reliquis x et y formatur, non explicite exhibeatur, sed per resolutionem demum aequationis inuentae definiri debet, cuius rei aliquot exempla euoluisse iuuabit.

Exemplum 1.

56. *Quaeratur eiusmodi functio* z *ipsarum* x *et* y, *vt sit* $\left(\frac{dz}{dx}\right) = \frac{y}{z}$.

Cum ergo sit $dz = \frac{y\,dx}{z} + q\,dy$ erit y pro constante sumendo

$$z\,dz = y\,dx \text{ et } \tfrac{1}{2}zz = xy + f:y.$$

Pro q inueniendo differentietur haec aequatio generaliter

$$z\,dz = y\,dx + x\,dy + dy f':y$$

eritque

$$q = \frac{x}{z} + \frac{1}{z}f':y,$$

quod idem per regulam datam reperitur. Nam ob $V = \frac{y}{z}$, erit $\int\frac{dz}{V} = \frac{zz}{2y}$ integrali ita sumto vt eua-

 nescat

nescat posito $z=0$; tum vero ob $\left(\frac{dV}{dy}\right)=\frac{1}{z}$ erit

$$\int\frac{dz}{VV}\left(\frac{dV}{dy}\right)=\int\frac{z\,dz}{yy}=\frac{zz}{2yy}$$

eadem integrationis lege obseruata. Hinc fit

$$dz=\frac{y\,dx}{z}+\frac{y\,dy}{z}\left(\frac{zz}{2yy}+f':y\right) \text{ et } q=\frac{z}{2y}+\frac{y}{z}f':y$$

quae expressio cum praecedente conuenit, ex comparatione enim fit

$$x+f':y=\frac{zz}{2y}+yf':y,$$

vnde x aequatur vt ante quantitati $\frac{zz}{2y}$ vna cum functione ipsius y. Hoc tantum notetur, quod ad consensum perfectum hic pro $f:y$ scribere debuissemus $yf:y$.

Exemplum 2.

57. *Quaeratur eiusmodi functio* z *binarum variabilium* x *et* y, *vt sit* $\left(\frac{dz}{dx}\right)=\frac{\sqrt{(yy-zz)}}{z}$.

Cum ergo sit

$$dz=\frac{dx\sqrt{(yy-zz)}}{z}+q\,dy,$$

sumta y constante fit

$$dx=\frac{z\,dz}{\sqrt{(yy-zz)}}, \text{ et integrando}$$

$$x=y-\sqrt{(yy-zz)}-f:y;$$

vnde vicissim differentiando oritur

$$dx=dy-\frac{y\,dy+z\,dz}{\sqrt{(yy-zz)}}-dyf':y \text{ seu}$$

$$dz=\frac{dx\sqrt{(yy-zz)}}{z}+dy\left(\frac{y}{z}-\frac{\sqrt{(yy-zz)}}{z}(1-f':y)\right).$$

Per

Per regulam autem datam ob $V = \frac{\sqrt{(yy-zz)}}{z}$, eſt $\int \frac{dz}{V} = y - \sqrt{(yy-zz)}$ integrali ita ſumto, vt euaneſcat poſito $z = 0$. Iam vero eſt

$$\left(\frac{dV}{dy}\right) = \frac{y}{z\sqrt{(yy-zz)}} \text{ et } \frac{1}{VV}\left(\frac{dV}{dy}\right) = \frac{yz}{(yy-zz)^{\frac{3}{2}}},$$

hinc

$$\int \frac{dz}{VV}\left(\frac{dV}{dy}\right) = \frac{y}{\sqrt{(yy-zz)}} - 1,$$

integrali eadem lege ſumto. Quocirca colligitur

$$q = \frac{\sqrt{(yy-zz)}}{z}\left(\frac{y}{\sqrt{(yy-zz)}} - 1 + f':y\right) = \frac{y}{z} - \frac{\sqrt{(yy-zz)}}{z}(1 - f':y)$$

prorſus vt ante.

Problema 8.

58. Si z ita debeat determinari per binas variabiles x et y, vt formula differentialis $\left(\frac{dz}{dx}\right) = p$ aequetur functioni cuipiam datae ipſarum x et z, quae ſit $= V$ definire in genere indolem functionis z per x et y.

Solutio.

Ponatur $dz = p\,dx + q\,dy$, et cum ſit $p = V$, ſumatur quantitas y conſtans, eritque $dz - V\,dx = 0$, quae aequatio duas tantum quantitates variabiles x et z continens, integrabilis reddetur ope cuiusdam multiplicatoris, qui ſit $= M$, ita vt $M\,dz - MV\,dx$ ſit differentiale verum cuiuſpiam functionis ipſarum x et z, quae functio ſit $= S$, quantitatem y non inuoluens.

voluens. Ex quo aequatio nostra integralis erit $S=f:y$, vnde indoles functionis z quemadmodum per x et y determinatur, innotescit. Differentiemus hanc aequationem sumto praeter x et z etiam y variabili, eritque

$$dS = Mdz - MVdx = dyf':y \text{ seu}$$
$$dz = Vdx + \frac{dy}{M}f':y \text{ ita vt sit } q = \frac{1}{M}f':y.$$

Coroll. 1.

59. Multiplicator etiam M formulam dz-Vdx integrabilem reddens, quantitatem y non continebit, quia in functione data V non inest y. Statim autem hoc multiplicatore inuento, valor ipsius $q=\frac{1}{M}f':y$ colligitur.

Coroll. 2.

60. Si formulae differentialis $Mdz-MVdx$ integrale fuerit S functio ipsarum x et z, pro solutione problematis habebimus $S=f:y$, vnde patet constantem, quam quis forte ad S adiicere voluerit, iam in functione arbitraria $f:y$ contineri.

Exemplum 1.

61. *Quaeratur eiusmodi functio z ipsarum x et y, vt sit* $\left(\frac{dz}{dx}\right)=\frac{nz}{x}$.

Posito $dz=\frac{nzdx}{x}+qdy$, sumto y constante erit $dz-\frac{nzdx}{x}=0$, quae aequatio per $\frac{1}{z}$ multiplicata

cata fit integrabilis, ita vt fit multiplicator $M = \frac{1}{z}$, hincque integrale $S = lz - lx^n$; ergo aequatio nostra integralis quaesita erit $l\frac{z}{x^n} = f:y$; vnde etiam $\frac{z}{x^n}$ aequabitur functioni cuicunque ipsius y, ita vt fit $z = x^n f:y$.

Exemplum 2.

62. *Quaeratur eiusmodi functio z binarum variabilium x et y, vt fit formula differentialis* $(\frac{dz}{dx}) = nx - z$.

Posito $dz = (nx - z)dx + qdy$, sumto y constante erit $dz + zdx - nxdx = 0$, quae ope multiplicatoris $M = e^x$ dat

$$S = e^x z - n\int e^x x dx = e^x z - ne^x x + ne^x;$$

vnde aequatio quaesitam relationem inter x, y et z exprimens est

$$e^x z - ne^x x + ne^x = f:y \text{ siue}$$

$$z = n(x-1) + e^{-x} f:y$$

tum vero erit

$$q = (\tfrac{dz}{dy}) = e^{-x} f':y.$$

Exemplum 3.

63. *Quaeratur eiusmodi functio z binarum variabilium x et y, vt fit formula differentialis* $(\frac{dz}{dx}) = \frac{xz}{xx+zz}$.

Ponatur ergo $dz = \frac{xzdx}{xx+zz} + qdy$ et posito y constante quaeratur integrale huius aequationis differentialis:

$$dz - \frac{xzdx}{xx+zz} = 0,$$

quae integrabilis redditur ope cuiusdam diuiforis, qui ob homogeneitatem reperitur fcribendo x et z loco differentialium dx et dz, ita vt hic diuifor fit:

$$z - \frac{xxz}{xx+zz} = \frac{z^3}{xx+zz},$$

hincque multiplicator $M = \frac{xx+zz}{z^3}$. Quare erit

$$dS = \frac{(xx+zz)dz}{z^3} - \frac{xdx}{zz}, \text{ ideoque}$$

$$S = \frac{-xx}{2zz} + lz,$$

vnde aequatio noftra quaefita erit

$$lz - \frac{xx}{2zz} = f:y \text{ et } q = \frac{z^3}{xx+zz} f':y$$

ex qua cum pofito $lz - \frac{xx}{2zz} = u$ fit $u = f:y$ etiam viciffim concludi poteft fore $y = f:u$.

Problema 9.

64. Si z ita debeat determinari per binas variabiles x et y, vt formula differentialis $\left(\frac{dz}{dx}\right)$ aequetur functioni cuipiam datae omnes tres variabiles x, y et z implicanti, quae fit $= V$, definire in genere indolem functionis z per x et y.

Solutio.

Cum fit $dz = Vdx + qdy$, fi fumamus y conftans, erit $dz = Vdx$ quae ergo aequatio duas tantum

tum

tum continet variabiles x et z, litteram autem y in functione V inuoluens. Dabitur ergo multiplicator M hanc aequationem integrabilem reddens, ita vt fit

$$M\,dz - MV\,dx = dS$$

vnde aequatio integralis relationem inter x, y et z exprimens erit

$$S = f:y$$

vbi S erit functio certa ipsarum x, y et z, fierique potest vt etiam M omnes has tres litteras comprehendat. Conuenit autem functioni S per integrationem inuentae valorem determinatum tribui, quoniam pars indeterminata in functione arbitraria $f:y$ includitur. Ponamus ergo S ita capi vt euanescat si ponatur $x = a$ et $z = c$.

Quod si hinc aequationis differentialis propositae alteram partem $q\,dy$ inuenire velimus, differentiemus functionem S sumto etiam y variabili sitque

$$dS = M\,dz - MV\,dx + Q\,dy = dy\,f':y$$

vbi cum sit

$$\left(\frac{dQ}{dz}\right) = \left(\frac{dM}{dy}\right) \text{ vel } \left(\frac{dQ}{dx}\right) = -\left(\frac{d.MV}{dy}\right)$$

erit sumto iterum y constante:

$$dQ = dz\left(\frac{dQ}{dz}\right) + dx\left(\frac{dQ}{dx}\right) = dz\left(\frac{dM}{dy}\right) - dx\left(\frac{d.MV}{dy}\right)$$

quae formula certo erit integrabilis. Capi autem Q eadem lege debet, qua S sumsimus, ita vt euanescat posito $x = a$ et $z = c$, atque inuenta hac quantitate Q, cum habeamus

$$dz = V\,dx - \frac{Q\,dy}{M} + \frac{dy}{M}f':y$$

erit $q = \left(\frac{dz}{dy}\right) = \frac{-Q + f':y}{M}$.

Haec determinatio isto nititur fundamento, quod si S fuerit eiusmodi functio ipsarum x, y et z, quae posito $x = a$ et $z = c$ euanescat, etiam formula differentialis $\left(\frac{dS}{dy}\right)$ eodem casu euanescat.

Coroll. 1.

65. Reducitur ergo resolutio huius problematis ad integrationem aequationis differentialis

$$dz - V\,dx = 0,$$

in qua quantitas y vt constans spectatur, etiamsi V contineat omnes tres litteras x, y et z. Dabitur ergo vtique multiplicator M, qui hanc aequationem integrabilem reddat, vt sit

$$M\,dz - MV\,dx = dS,$$

existente S certa quadam functione ipsarum x, y et z.

Coroll. 2.

66. Inuento autem hoc multiplicatore M indeque integrali S quantitas z ita per binas variabiles x et y definietur, vt sit $S = f:y$ vbi $f:y$ denotat functionem quamcunque ipsius y siue continuam siue etiam discontinuam, ob cuius naturam integratio pro completa est habenda.

Coroll. 3.

67. Cum hoc modo relatio inter z, x, y fuerit definita, erit ea ita differentiata, vt omnes tres litterae

litterae x, y et z variabiles ſumantur:

$$dz = V\,dx + \left(\frac{f':y - Q}{M}\right)dy,$$

vbi quantitas Q ex ſuo differentiali

$$dQ = dz\left(\frac{dM}{dy}\right) - dx\left(\frac{d.MV}{dy}\right)$$

definiri debet, ſumta y conſtante, integrationem ita temperando, vt ſi S euaneſcat caſu $x = a$ et $z = c$, etiam Q eodem caſu euaneſcat.

Scholion.

68. Hic ergo ad inſigne hoc Theorema deducimur:

Quod ſi fuerit S *eiusmodi functio ipſarum* x, y *et* z, *quae euaneſcat ponendo* $x = a$ *et* $z = c$, *tum etiam pro eadem poſitione formulam* $\left(\frac{dS}{dy}\right)$ *eſſe euanituram.*

Veluti ſi fuerit

$$S = Axx + Bxyz + Czz - Daa - Bacy - Ccc$$

erit $\left(\frac{dS}{dy}\right) = Bxz - Bac$,

quarum vtraque expreſſio caſu $x = a$ et $z = c$ euaneſcit. Pluribus autem huiusmodi exemplis euolutis veritas Theorematis ita patet, vt demonſtratio ſolennis non deſideretur. Interim huiusmodi functio ſemper quantitates ſolam y continentes a reliquis ſeparando ita euolui poteſt, vt in talem formam transmutetur:

$$S = PY + QY' + RY'' \text{ etc.}$$

 vbi

vbi per hypothesin P, Q, R etc. sunt functiones ipsarum x et z tantum, et tales quidem quae ponendo $x=a$ et $z=c$ singulae euanescant. Hinc iam perspicuum est fore

$$\left(\frac{dS}{dy}\right)=P.\frac{dV}{dy}+Q.\frac{dV'}{dy}+R.\frac{dV''}{dy} \text{ etc.}$$

quae forma manifesto sub iisdem conditionibus euanescit. Quomodocunque autem functio S hac indole praedita fuerit complicata tam formulis irrationalibus quam transcendentibus, eam semper in eiusmodi formam euoluere licet, quae etsi in infinitum progrediatur, haec demonstratio tamen vim suam retinet.

Exemplum 1.

69. *Quaeratur eiusmodi functio* z *duarum variabilium* x *et* y *vt sit formula differentialis* $\left(\frac{dz}{dx}\right)=\frac{xz}{ay}$.

Ponamus ergo $dz=\frac{xzdx}{ay}+qdy$, et sumta y constante habebitur aequatio $dz-\frac{xzdx}{ay}=0$, vt sit $V=\frac{xz}{ay}$, et multiplicator erit $M=\frac{1}{z}$; vnde fit

$$S=l\frac{z}{c}-\frac{xx-aa}{2ay}$$

et aequatio integralis completa functionem z determinans erit

$$l\frac{z}{c}+\frac{aa-xx}{2ay}=f:y.$$

Porro ad quantitatem q inueniendam, ob $M=\frac{1}{z}$ et $MV=\frac{x}{ay}$, erit $dQ=0$ et $Q=0$; vnde fit $q=zf':y$. Hic idem autem valor ex differentiatione aequa-

aequationis inuentae eruitur, quae praebet

$\frac{dz}{z}-\frac{xdx}{ay}=dyf':y$ ideoque

$dz=\frac{xzdx}{ay}+zdyf':y$, ita vt fit $q=zf':y$.

Exemplum 2.

70. *Quaeratur binarum variabilium x et y eiusmodi functio z vt fit* $\left(\frac{dz}{dx}\right)=\frac{y}{x+z}$.

Cum fit $V=\frac{y}{x+z}$, habebitur fumto y conftante haec aequatio

$$dz-\frac{ydx}{x+z}=0,$$

ad cuius multiplicatorem inueniendum multiplicetur primo per $x+z$ vt prodeat

$$xdz+zdz-ydx=0 \text{ feu } dx-\frac{xdz}{y}=\frac{zdz}{y},$$

quae multiplicata per $e^{-\frac{z}{y}}$ integrabilis euadit, proditque

$$e^{-\frac{z}{y}}x=\int e^{-\frac{z}{y}}\frac{zdz}{y}=-e^{-\frac{z}{y}}z+\int e^{-\frac{z}{y}}dz$$

hincque

$$e^{-\frac{z}{y}}x=-e^{-\frac{z}{y}}z-ye^{-\frac{z}{y}}+C.$$

Quocirca erit multiplicator

$$M=(x+z).-\tfrac{1}{y}.e^{-\frac{z}{y}}=-\frac{(x+z)}{y}e^{-\frac{z}{y}} \text{ et}$$

$$S=e^{-\frac{z}{y}}(x+z+y)-e^{-\frac{c}{y}}(a+c+y)$$

ex

ex quo aequatio integralis completa erit

$$e^{-\frac{z}{y}}(x+z+y)-e^{-\frac{c}{y}}(a+c+y)=f:y.$$

Nunc porro cum sit $MV=-e^{-\frac{z}{y}}$ erit

$$\left(\frac{dM}{dy}\right)=e^{-\frac{z}{y}}\left(\frac{x+z}{yy}-\frac{z(x+z)}{y^3}\right)=e^{-\frac{z}{y}}(x+z)\left(\frac{1}{yy}-\frac{z}{y^3}\right)$$

et $\left(\frac{d.MV}{dy}\right)=-e^{-\frac{z}{y}}\cdot\frac{z}{yy}$ hincque

$$dQ=e^{-\frac{z}{y}}\left(dz(x+z)\left(\frac{1}{yy}-\frac{z}{y^3}\right)+\frac{z\,dz}{yy}\right)$$

sumto y constante, vnde integrando obtinebitur:

$$Q=e^{-\frac{z}{y}}\left(\frac{xz}{yy}+1+\frac{z}{y}+\frac{zz}{yy}\right)-e^{-\frac{c}{y}}\left(\frac{ac}{yy}+1+\frac{c}{y}+\frac{cc}{yy}\right)$$

hinc

$$q=\frac{z}{y}+\frac{y+z}{x+z}-e^{\frac{z-c}{y}}\left(\frac{ac+cc+cy+yy}{y(x+z)}\right)-\frac{y}{x+z}e^{\frac{z}{y}}f':y$$

ita vt sit

$$dz=\frac{y\,dx}{x+z}+q\,dy.$$

Aequatio autem inuenta si differentietur dat:

$$-e^{-\frac{z}{y}}\frac{(x+z)dz}{y}+e^{-\frac{z}{y}}dx+e^{-\frac{z}{y}}dy\left(1+\frac{z}{y}+\frac{xz}{yy}+\frac{zz}{yy}\right)$$
$$-e^{-\frac{c}{y}}dy\left(1+\frac{c}{y}+c\frac{(a+c)}{yy}\right)=dy\,f':y$$

vnde idem prorsus valor pro q concluditur.

Exem-

Exemplum 3.

71. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z *vt sit* $\left(\frac{dz}{dx}\right) = \frac{yy+zz}{yy+xx}$.

Posito $dz = \frac{yy+zz}{yy+xx}dx + q\,dy$, sumatur quantitas y constans et cum sit $dz - \frac{(yy+zz)dx}{yy+xx} = 0$, euidens est multiplicatorem idoneum esse $M = \frac{y}{yy+zz}$, vnde cum sit $\frac{y\,dz}{yy+zz} - \frac{y\,dx}{yy+xx} = 0$ erit per integrationem

$$S = \text{A tang.}\frac{z}{y} - \text{A tang.}\frac{x}{y} + C = \text{A tang.}\frac{yz-yx}{yy+xz} - \text{A tang.}\frac{(c-a)y}{ac+yy}$$

et functio quaesita z hac aequatione definietur:

$$\text{A tang.}\frac{y(z-x)}{yy+xz} - \text{A tang.}\frac{(c-a)y}{ac+yy} = f:y.$$

Cum porro sit $MV = \frac{y}{yy+xx}$ erit

$$\left(\frac{dM}{dy}\right) = \frac{zz-yy}{(yy+zz)^2} \text{ et } \left(\frac{d.MV}{dy}\right) = \frac{xx-yy}{(yy+xx)^2},$$

hincque

$$dQ = \frac{(zz-yy)dz}{(yy+zz)^2} - \frac{(xx-yy)dx}{(yy+xx)^2}$$

sumto y constante. Ergo

$$Q = \frac{-z}{yy+zz} + \frac{x}{yy+xx} + \frac{c}{yy+cc} - \frac{a}{yy+aa}$$

et $q = \frac{-Q+f':y}{M}$, qui idem valor etiam ex differentiatione prodit.

Ceterum cum constantes a et c pro lubitu accipi queant, sumtis iis nihilo aequalibus, seu saltem $c = a$, erit aequatio integralis:

$$\text{A tang.}\frac{y(z-x)}{yy+xz} = f:y,$$

 vnde

vnde erit etiam

$$\frac{y(z-x)}{yy+xz} = \text{funct.}\, y \quad \text{et} \quad \frac{yy+xz}{z-x} = \text{funct.}\, y\,;$$

quae functio si dicatur Y habebitur:

$$z = \frac{yy+x\mathrm{Y}}{\mathrm{Y}-x}.$$

Scholion.

72. Vix opus est notari saepe fieri posse, vt solutio huiusmodi quaestionum superet vires analyseos, quando scilicet aequatio differentialis resoluenda artificiis adhuc cognitis integrari nequit. Veluti si proponatur casus $\left(\frac{dz}{dx}\right) = \frac{yy}{xx+zz}$, vnde sumto y constante fieri debet $yydx = xxdz + zzdz$, cuius integrationem nondum expedire licet. Interim quia integrale per seriem exhiberi potest, modo id fiat complete, etiam solutio per seriem obtinebitur. Posito scilicet $x = \frac{-yydu}{udz}$, et sumto elemento dz constante, oritur haec aequatio differentio-differentialis

$$y^4 ddu + uzzdz^2 = 0$$

vnde per series integrando reperitur

$$u = \mathrm{A}\left(1 - \frac{z^4}{3\cdot 4 y^4} + \frac{z^8}{3\cdot 4\cdot 7\cdot 8 y^8} - \text{etc.}\right) + \mathrm{B}z\left(1 - \frac{z^4}{4\cdot 5 y^4} + \frac{z^8}{4\cdot 5\cdot 8\cdot 9 y^8} - \text{etc.}\right)$$

vbi pro A et B functiones quaecunque ipsius y accipi possunt. Quare si ponatur $\frac{\mathrm{A}}{\mathrm{B}} = f:y$ erit

$$x = \frac{yyf:y\left(\frac{z^3}{3y^4} - \frac{z^7}{3\cdot 4\cdot 7 y^8} + \text{etc.}\right) - yy\left(1 - \frac{z^4}{4y^4} + \frac{z^8}{4\cdot 5\cdot 8 y^8} - \text{etc.}\right)}{f:y\left(1 - \frac{z^4}{3\cdot 4 y^4} + \frac{z^8}{3\cdot 4\cdot 7\cdot 8 y^8} - \text{etc.}\right) + z\left(1 - \frac{z^4}{4\cdot 5 y^4} + \frac{z^8}{4\cdot 5\cdot 8\cdot 9 y^8} - \text{etc.}\right)}$$

qua aequatione functio quaesita z, per binas variabiles x et y generalissime exprimitur. Quoniam ergo metho-

methodos aperuimus aequationes differentiales quascunque per approximationes integrandi, idque complete; his methodis in subsidium vocandis, omnia problemata huc pertinentia saltem per approximationem resolui poterunt. Ceterum in hac parte Analyseos sublimiori resolutionem aequationum differentialium ad priorem partem Analysis pertinentium pro concessa assumere possumus, omnino vti, quo longius in Analysi progredimur, ea semper quae ad partes praecedentes pertinent; etiamsi non penitus sunt euoluta, tanquam confecta spectare solemus.

CAPVT III.

DE RESOLVTIONE AEQVATIONVM QVIBVS BINARVM FORMVLARVM DIFFERENTIALIVM ALTERA PER ALTERAM VTCVNQVE DATVR.

Problema 10.

73.

Si z eiusmodi esse debeat functio binarum variabilium x et y, vt formulae differentiales $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ inter se fiant aequales, indolem istius functionis in genere determinare.

Solutio.

Ponatur $\left(\frac{dz}{dx}\right) = p$ et $\left(\frac{dz}{dy}\right) = q$, vt sit $dz = pdx + qdy$, haecque formula $pdx + qdy$ integrationem sponte admittat. Quoniam igitur requiritur vt sit $q = p$, erit $dz = p(dx + dy)$, et posito $x + y = u$, fiet $dz = pdu$, quae formula cum debeat esse per se integrabilis, necesse est vt p sit functio quantitatis variabilis u, nullam praeterea aliam variabilem inuoluens; hincque integrando ipsa quantitas $z = \int pdu$ aequa-

aequabitur functioni ipsius u, seu prodibit $z=f:u$, quae functio omnino arbitrio nostro relinquitur, ita vt pro z functio quaecunque ipsius u siue continua siue etiam discontinua assumta problemati satisfaciat. Quare cum sit $u=x+y$, erit pro solutione nostri problematis $z=f:(x+y)$. Quae forma, quo facilius appareat, quomodo conditioni praescriptae satisfaciat, sit $d.f:u=duf':u$, ideoque ob $u=x+y$ erit

$$dz=(dx+dy)f':(x+y)=dxf':(x+y)+dyf'(x+y)$$

ideoque et

$$\left(\frac{dz}{dx}\right)=p=f':(x+y) \text{ et } \left(\frac{dz}{dy}\right)=q=f':(x+y)$$

ac propterea $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$ seu $q=p$, omnino vti problema postulat.

Coroll. 1

74. Quaecunque ergo functio quantitatis $x+y$ formetur, ea pro z assumta praescriptam habebit proprietatem, vt sit $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$. Talem autem functionem indicamus signo $f:(x+y)$, ita vt sit $z=f:(x+y)$.

Coroll. 2.

75. Geometrice haec solutio ita referri potest. Descripta super axe linea curua quacunque siue regulari siue irregulari, si abscissa exprimatur per $x+y$, applicata semper idoneum valorem pro functione z exhibebit.

Coroll. 3.

76. Vniuerſalitas huius ſolutionis per integrationem erutae in hoc conſiſtit, quod quantitatis $x+y$ functionem qualemcunque ſiue continuam ſiue etiam diſcontinuam pro z inuenerimus; quippe quae conditioni problematis ſemper ſatisfacit.

Scholion 1.

77. Fundamentum ſolutionis hoc nititur principio, quod formula differentialis pdu integrabilis eſſe nequeat, niſi quantitas p ſit functio ipſius u, vel viciſſim u functio ipſius p, ita vt nulla alia variabilis in computum ingrediatur. Quin etiam qualiscunque fuerit p functio ipſius u integrale, niſi actu exhiberi, ſemper tamen concipi poteſt; ſi enim u denotet abſciſſam, et p applicatam curuae cuiuscunque ſiue regularis ſiue irregularis, qua ratione vtique functio quaecunque ipſius u in ſenſu latiſſimo repraeſentari poteſt, eius curuae area $\int pdu$ praebet valorem formulae integralis $\int pdu$, quae iterum vt functio ipſius u ſpectari poteſt; ex quo viciſſim functio quaecunque ipſius u naturam formulae integralis $\int pdu$ exhaurit. Quod autem functio quaecunque quantitatis $x+y$ pro z aſſumta ſatisfaciat conditioni, vt in differentiali $dz=pdx+qdy$, fiat $p=q$ ſeu $(\frac{dz}{dx})=(\frac{dz}{dy})$, ita per ſe eſt perſpicuum, vt illuſtratione per exempla non egeat. Si enim verbi gratia ponatur:

$$z=aa+b(x+y)+(x+y)^2=aa+bx+by+xx+2xy+yy$$

erit

erit differentiando:

$\therefore \left(\frac{dz}{dx}\right) = b + 2x + 2y$ et $\left(\frac{dz}{dy}\right) = b + 2x + 2y$,

qui valores inter se vtique sunt aequales.

Scholion 2.

78. Cum z sit functio binarum variabilium x et y, ac ponatur $dz = pdx + qdy$, vt sit

$$\left(\frac{dz}{dx}\right) = p \text{ et } \left(\frac{dz}{dy}\right) = q,$$

in hoc capite eiusmodi quaestiones euoluere est propositum in quibus aequatio quaecunque inter p et q praescribitur, in quam reliquarum variabilium x, y et z nulla ingrediatur. Proposita ergo aequatione quacunque inter binas formulas p et q et constantes, quaeri oportet indolem functionis z binarum variabilium x et y, vt formulis inde per differentiationem natis $p = \left(\frac{dz}{dx}\right)$ et $q = \left(\frac{dz}{dy}\right)$ praescripta illa conditio conueniat. Quam tractationem quidem exorsi sumus ab exemplo simplicissimo $p = q$, cuius solutio etiam ope principii modo expositi, confici potest. At vero idem principium sufficit problemati sequenti latius patenti resoluendo.

Problema 11.

79. Si z eiusmodi esse debeat functio binarum variabilium x et y, vt fiat $\alpha\left(\frac{dz}{dx}\right) + \beta\left(\frac{dz}{dy}\right) = \gamma$, indolem istius functionis z in genere definire.

Solutio.

Solutio.

Posito $dz = p\,dx + q\,dy$ requiritur vt sit $\alpha p + \beta q = \gamma$. Hinc cum sit $q = \frac{\gamma - \alpha p}{\beta}$ erit

$$dz = p\,dx + \frac{(\gamma - \alpha p)}{\beta} dy \text{ seu}$$

$$dz = \frac{\gamma}{\beta} dy + \frac{p}{\beta}(\beta\,dx - \alpha\,dy)$$

quam formulam integrabilem esse oportet. Cum autem pars $\frac{\gamma}{\beta} dy$ per se sit integrabilis, altera pars etiam integrabilis sit necesse est, vnde posito $\beta x - \alpha y = u$ vt altera pars fiat $\frac{p}{\beta} du$, euidens est p functionem esse debere ipsius u, indeque etiam integrale proditurum esse functionem ipsius $u = \beta x - \alpha y$. Quare ponamus

$$\int p(\beta\,dx - \alpha\,dy) = f:(\beta x - \alpha y),$$

eritque

$$z = \frac{\gamma}{\beta} y + \frac{1}{\beta} f:(\beta x - \alpha y)$$

seu aequatio quaesita indolem functionis z determinans erit

$$\beta z = \gamma y + f:(\beta x - \alpha y)$$

denotante signo $f:$ functionem quamcunque siue continuam siue discontinuam formulae suffixae $\beta x - \alpha y$. Atque indicando formulae $f:u$ differentiale per $du f':u$ erit

$$p = f':(\beta x - \alpha y) \text{ et } q = \frac{\gamma}{\beta} - \frac{\alpha}{\beta} f':(\beta x - \alpha y)$$

vnde manifesto resultat $\alpha p + \beta q = \gamma$.

Coroll. 1.

Coroll. 1.

80. Eodem folutio redit, fi pro p eius valorem $p = \frac{\gamma - \beta q}{\alpha}$ fubftituamus, vnde fit

$$dz = \frac{\gamma}{\alpha} dx + \frac{q}{\alpha}(\alpha dy - \beta dx),$$

hincque eodem modo

$$z = \frac{\gamma x}{\alpha} + \frac{1}{\alpha} f:(\alpha y - \beta x).$$

Etfi enim haec forma a praecedente differre videtur, tamen facile eo reducitur, ponendo ibi

$$f:(\beta x - \alpha y) = \frac{\gamma(\beta x - \alpha y)}{\alpha} + \frac{\beta}{\alpha} \Phi:(\alpha y - \beta x),$$

quae forma vtique eft functio ipfius $\beta x - \alpha y$.

Coroll. 2.

81. Si ergo in forma $dz = pdx + qdy$ debeat effe $p + q = 1$ ob $\alpha = 1$, $\beta = 1$ et $\gamma = 1$, folutio huc redit, vt fiat:

$$z = y + f:(x - y).$$

Conftructa ergo curua quacunque, fi abfciffae $x - y$ refpondeat applicata v, erit $z = y + v$.

Scholion.

82. Si alia proponatur relatio inter p et q, eadem methodo folutionem obtinere non licet; fed alio principio vti conuenit; cuius quidem veritas ex primis calculi integralis elementis eft manifefta.

Notari scilicet oportet esse

$$\int p\,dx = px - \int x\,dp$$

similique modo

$$\int q\,dy = qy - \int y\,dq,$$

ita vt cum sit

$$z = \int (p\,dx + q\,dy)$$

futurum sit

$$z = px + qy - \int (x\,dp + y\,dq).$$

Quomodo autem hoc principium ad solutionem huiusmodi quaestionum, quae ad hoc caput sint referendae, applicandum sit, in sequentibus problematibus docebitur.

Problema 12.

83. Si z eiusmodi esse debeat functio binarum variabilium x et y, vt posito $dz = p\,dx + q\,dy$, fiat $pq = 1$, indolem istius functionis z in genere definire.

Solutio.

Ex principio ante stabilito notemus fore

$$z = px + qy - \int (x\,dp + y\,dq).$$

Cum iam ob $pq = 1$ sit $q = \frac{1}{p}$ erit

$$z = px + \frac{y}{p} - \int \left(x\,dp - \frac{y\,dp}{pp}\right).$$

Integrabilis ergo esse debet haec forma $\int \left(x - \frac{y}{pp}\right) dp$, at in genere formula $\int u\,dp$ integrationem non admittit

mittit nisi sit u functio ipsius p. Quare in nostro casu necesse est sit quantitas $x - \frac{y}{pp}$ functio ipsius p tantum, vnde etiam integrale $\int dp(x - \frac{y}{pp})$ erit functio ipsius p tantum, quae si indicetur per $f:p$ eiusque differentiale per $dpf':p$ erit

$$z = px + \frac{y}{p} - f:p \quad \text{et} \quad x - \frac{y}{pp} = f':p.$$

Quare ad problema nostrum soluendum, noua variabilis p introduci debet, ex qua cum altera y coniuncta binae reliquae x et z determinentur. Sumta scilicet variabili p eiusque functione quacunque $f:p$, indeque per differentiationem deriuata $f':p$, capiatur primo

$$x = \frac{y}{pp} + f':p, \text{ indeque erit}$$

$$z = \frac{2y}{p} + pf':p - f:p,$$

quae est solutio problematis quaesita generalis.

Coroll. 1.

84. Hic igitur functio quaesita z per ipsas variabiles x et y explicite euolui nequit; propterea quod quantitatem p ex aequatione $x - \frac{y}{pp} = f':p$ in genere per x et y definire non licet.

Coroll. 2.

85. Nihilo vero minus solutio pro idonea et completa est habenda, quoniam introducendo nouam variabilem p ex binis y et p a se inuicem non pendentibus ambae reliquae x et z definiuntur.

Coroll. 3.

86. Si sumamus $f':p = \alpha + \frac{\beta}{pp}$, erit

$$f:p = \alpha p - \frac{\beta}{p} \text{ et } (x-\alpha) = \frac{\beta + y}{pp},$$

hinc $p = \sqrt{\frac{\beta + y}{x - \alpha}}$; vnde functio quaesita z ita se habebit

$$z = \frac{2y\sqrt{(x-\alpha)}}{\sqrt{(\beta+y)}} + \frac{\alpha y + \beta x}{\sqrt{(x-\alpha)(\beta+y)}} - \frac{\alpha y + \beta x - 2\alpha\beta}{\sqrt{(x-\alpha)(\beta+y)}}$$

seu $z = 2\sqrt{(x-\alpha)(y+\beta)}$, quae est solutio particularis, et simplicissima est $z = 2\sqrt{xy}$.

Scholion 1.

87. Quemadmodum solutio huius problematis ex alio principio est deducta, ita etiam forma solutionis a praecedentibus discrepat, quod hic aequationem inter x, y et z explicitam exhibere non liceat, sed noua variabilis p introducatur. Cum igitur ante vna aequatio inter ternas variabiles x, y et z solutionem, continuisset; nunc accedente noua variabili p, solutio geminam aequationem inter has quatuor variabiles postulat, sicque pro nostro casu inuenimus:

$$z = px + \frac{y}{p} - f:p \text{ et } x - \frac{y}{pp} = f':p$$

existente

$$d.f:p = dpf':p,$$

vbi functionis signum indefinitum f: quod etiam functiones discontinuas admittit, vniuersalitatem solutionis praestat. Quod si hinc litteram p eliminare

liceret,

liceret, aequatio euoluta inter x, y et z obtineretur, haec autem eliminatio succedit, quoties pro $f:p$ functio algebraica ipsius p assumitur; in genere autem nullo modo sperari potest. Nihilo vero minus ope curuae pro lubitu assumtae problema construi potest; sumta enim curua quacunque siue regulari siue irregulari, ponatur abscissa $=p$, sitque applicata $f':p=r$, erit $f:p=\int rdp$ area eius curuae, quae si dicatur $=s$, aequationes binae

$$x-\frac{y}{pp}=r \text{ et } z=px+\frac{y}{p}-s$$

solutionem completam problematis praebebunt. Scilicet sumto pro x valore quocunque, erit $y=pp(x-r)$ hincque fit $z=2px-pr-s$, in qua solutione nihil ad praxin spectans desiderari potest. Hinc patet etiam fortasse fieri posse, vt duae nouae variabiles sint introducendae, ac tum solutio tribus aequationibus contineatur; neque etiam tum quicquam deerit ad vsum practicum.

Scholion 2.

88. Cum pro formula $dz=pdx+qdy$ requiratur vt sit $pq=1$, introducendo angulum indefinitum Φ alia solutio concinnior elici potest. Posito enim $p=\text{tang.}\Phi$ erit $q=\cot.\Phi$, et ob $dz=dx\,\text{tang.}\Phi+dy\cot.\Phi$, fiet per reductionem supra indicatam

$$z=x\,\text{tang.}\Phi+y\cot.\Phi-\int d\Phi\left(\frac{x}{\cos.\Phi^2}-\frac{y}{\sin.\Phi^2}\right)$$

vnde patet formulam $\frac{x}{\cos.\Phi^2}-\frac{y}{\sin.\Phi^2}$ esse debere fun-

 ctionem

ctionem ipsius Φ, quae si ponatur $f':\Phi$, et formula integralis

$$\int d\Phi . f':\Phi = f:\Phi$$

binae aequationes solutionem continentes erunt:

$$\frac{x}{\cos.\Phi^2} - \frac{y}{\sin.\Phi^2} = f':\Phi \quad \text{et} \quad z = x\,\text{tang.}\Phi + y\cot.\Phi - f:\Phi$$

vnde iam pro lubitu x vel y eliminare licet. Quin etiam vtramque eliminare possumus, ac per binas variabiles z et Φ binae reliquae x et y ita exprimentur:

$$x = \tfrac{1}{2}z\cot.\Phi + \tfrac{1}{2}\cot.\Phi.f:\Phi + \tfrac{1}{2}\cos.\Phi^2.f':\Phi$$

$$y = \tfrac{1}{2}z\,\text{tang.}\Phi + \tfrac{1}{2}\text{tang.}\Phi.f:\Phi - \tfrac{1}{2}\sin.\Phi^2.f':\Phi.$$

Quodsi igitur hinc differentialia capiantur, ac ponatur $dy = 0$, ex posteriori dabitur, relatio inter dz et $d\Phi$, vnde si ipsius $d\Phi$ valor in priori substituatur, necesse est prodeat $dz = dx\,\text{tang.}\Phi$; simili autem modo si ponatur $dx = 0$, ex altera orietur $dz = dy\cot.\Phi$.

Problema 13.

89. Si z eiusmodi esse debeat functio binarum variabilium x et y, vt posito $dz = pdx + qdy$ fiat $pp + qq = 1$, indolem istius functionis z in genere inuestigare.

Solutio.

Cum per reductionem fiat

$$z = px + qy - \int(xdp + ydq)$$

vt

vt irrationalia euitemus, ponamus

$$p = \frac{1-rr}{1+rr} \text{ et } q = \frac{2r}{1+rr}$$

fiquidem hinc fit $pp+qq=1$. Erit autem

$$dp = \frac{-4rdr}{(1+rr)^2} \text{ et } dq = \frac{2dr(1-rr)}{(1+rr)^2}$$

hincque fit:

$$z = \frac{(1-rr)x+2ry}{1+rr} + 2\int \frac{2xrdr - ydr(1-rr)}{(1+rr)^2}$$

quae forma integralis cum fit functio ipfius r, ftatuatur ea $=f:r$, eiusque differentiale $=drf':r$, ex quo obtinebimus

$$\frac{2xr-y(1-rr)}{(1+rr)^2} = f':r \text{ et}$$

$$z = \frac{(1-rr)x+2ry}{1+rr} + 2f:r.$$

Vnde fi eliciamus

$$x = \frac{(1-rr)y}{2r} + \frac{(1+rr)^2}{2r} f':r \text{ erit}$$

$$z = \frac{(1+rr)y}{2r} + \frac{1-r^4}{2r} f':r + 2f:r.$$

Coroll. 1.

90. Si irrationalitatem non pertimefcamus ob $q=\sqrt{(1-pp)}$ et $dq = \frac{-pdp}{\sqrt{(1-pp)}}$ erit

$$z = px + y\sqrt{(1-pp)} - \int dp\left(x - \frac{py}{\sqrt{(1-pp)}}\right).$$

Pofito ergo $z = px + y\sqrt{(1-pp)} - f:p$ erit

$$x - \frac{py}{\sqrt{(1-pp)}} = f':p.$$

Coroll. 2.

Coroll. 2.

91. Solutio simplicissima sine dubio prodit sumendo $f:p=0$, vnde cum sit $x=\frac{py}{\sqrt{(1-pp)}}$, erit

$$p=\frac{x}{\sqrt{(xx+yy)}} \text{ et } \sqrt{(1-pp)}=\frac{y}{\sqrt{(xx+yy)}},$$

hincque

$$z=\frac{xx+yy}{\sqrt{(xx+yy)}}=\sqrt{(xx+yy)}.$$

Ex quo valore fit

$$\left(\frac{dz}{dx}\right)=p=\frac{x}{\sqrt{(xx+yy)}} \text{ et } \left(\frac{dz}{dy}\right)=\frac{y}{\sqrt{(xx+yy)}}=q,$$

ideoque $pp+qq=1$.

Coroll. 3.

92. Si ponamus $p=\sin.\Phi$ erit $q=\cos.\Phi$, hincque

$$z=x\sin.\Phi+y\cos.\Phi-\int d\Phi(x\cos.\Phi-y\sin.\Phi),$$

erit hoc integrale $=f:\Phi$, eiusque differentiale $d\Phi f':\Phi$. Ex quo habebimus

$$z=x\sin.\Phi+y\cos.\Phi-f:\Phi \text{ et } x\cos.\Phi-y\sin.\Phi=f':\Phi.$$

Problema 14.

93. Si z eiusmodi esse debeat functio binarum variabilium x et y, vt posito $dz=pdx+qdy$, quantitas q aequetur functioni datae ipsius p, indolem huius functionis z in genere inuestigare.

Solutio.

Solutio.

Cum q ſit functio data ipſius p, ponatur $dq = r dp$ erit etiam r functio data ipſius p. Aequatio ergo noſtra generalis ſolutionem ſuppeditans induet hanc formam

$$z = px + qy - \int dp(x + ry)$$

vnde patet integrale $\int dp(x + ry)$ fore functionem ipſius p, quae ſi generatim per $f:p$ exponatur, eiusque differentiale per $dpf':p$, habebimus:

$$z = px + qy - f:p \text{ et } x + ry = f':p$$

quae duae aequationes ſolutionem problematis vniuerſaliſſime complectuntur, ſiquidem $f:p$ functionem quamcunque ipſius p ſiue continuam ſiue diſcontinuam denotare poteſt.

Coroll. 1.

94. Cum ſit q functio data ipſius p, indeque $r = \frac{dq}{dp}$, ſi functio indefinita ipſius p ponatur $f:p = P$, ob $f'p = \frac{dP}{dp}$, ſolutio his aequationibus continebitur

$$z = px + qy - P \text{ et } x dp + y dq = dP.$$

Coroll. 2.

95. Si ad conſtructionem vtamur curua quacunque in qua ſi abſciſſa capiatur $= p$, applicata ſit $= f':p$ area eius curuae dabit valorem ipſius $f:p$.

Sin autem applicata indicetur per $f:p$, tum $f':p$ exprimet tangentem anguli, quem tangens curuae faciet cum axe.

Scholion.

96. Duplici ergo modo curua quaecunque ad libitum descripta siue sit continua seu aequatione quapiam analytica contenta, siue libero manus ductu vtcunque delineata, ad constructionem problematis adhiberi potest. Vel enim abscissa per p indicata, applicata sumi potest ad $f:p$ vel ad $f':p$ exprimendum, nec facile dici potest, vtrum ad praxin commodius sit futurum? Vbi autem huiusmodi problemata realiter occurrunt, reliquae circumstantiae solutionem determinare solent, vnde pro quouis casu constructio maxime idonea facile colligetur. Problemata autem mechanica hanc calculi integralis partem postulantia semper ad formulas differentiales secundi altiorumque ordinum deducunt, quarum resolutio ne suscipi quidem posse ante videtur, quam methodus pro formulis differentialibus primi gradus fuerit patefacta. Hactenus quidem problemata proposita absolute resoluere licuit; nunc autem quando conditio praescripta relationem formularum $(\frac{dz}{dx})$ et $(\frac{dz}{dy})$ per reliquas variabiles x, y et z definit, negotium in genere non amplius succedit, nisi relatio praescripta vnicam tantum variabilem cum binis formulis differentialibus coniungat.

CAPVT IV.

DE RESOLVTIONE AEQVATIONVM QVIBVS RELATIO INTER BINAS FORMVLAS DIFFERENTIALES ET VNICAM TRIVM QVANTITATVM VARIABILIVM PROPONITVR.

Problema 15.

97.

Si z eiusmodi esse debeat functio binarum variabilium x et y vt posito $dz = pdx + qdy$, sit $q = \frac{px}{a}$ indolem huius functionis in genere inuestigare.

Solutio.

Cum sit

$$dz = pdx + \frac{pxdy}{a} = px\left(\frac{dx}{x} + \frac{dy}{a}\right)$$

haecque formula esse debeat integrabilis, necesse est vt px ac proinde etiam z sit functio quantitatis $lx + \frac{y}{a}$. Quare solutio nostri problematis in genere ita se habebit, vt sit

$$z = f:(lx + \tfrac{y}{a}) \text{ et } px = f':(lx + \tfrac{y}{a})$$

L 2 sumen-

sumendo scilicet perpetuo $d.f:u = du f':u$. Hinc autem erit

$$p = \tfrac{1}{x} f':(lx + \tfrac{y}{a}) \text{ et } q = \tfrac{1}{a} f':(lx + \tfrac{y}{a}),$$

sicque $q = \frac{px}{a}$ omnino vti requiritur.

Coroll. 1.

98. Cum sit

$$z = px - \int x dp + \int \frac{pxdy}{a} = px + \int px\left(\frac{dy}{a} - \frac{dp}{p}\right),$$

hinc alia solutio deduci potest. Si enim ponamus

$$\int px\left(\frac{dy}{a} - \frac{dp}{p}\right) = f:\left(\frac{y}{a} - lp\right) \text{ erit } px = f':\left(\frac{y}{a} - lp\right)$$

indeque

$$z = f':\left(\frac{y}{a} - lp\right) + f:\left(\frac{y}{a} - lp\right).$$

Coroll. 2.

99. Hac ergo solutione noua introducitur variabilis p, ex qua cum y coniuncta definitur primo

$$x = \tfrac{1}{p} f':\left(\tfrac{y}{a} - lp\right)$$

tum vero ipsa functio quaesita

$$z = px + f:\left(\tfrac{y}{a} - lp\right).$$

Huic autem solutioni praecedens sine dubio antecellit, cum illa quantitatem z immediate per x et y exprimat.

Scho-

Scholion.

100. Quo has duas ſolutiones inter ſe comparare queamus quoniam functio arbitraria in vtraque diuerſae eſt indolis, etiam charactere diuerſo vtamur. Cum igitur prima praebeat:

$$z = f : \left(\frac{y}{a} + lx\right) \text{ et } px = f' : \left(\frac{y}{a} + lx\right)$$

altera vero:

$$z = F : \left(\frac{y}{a} - lp\right) + F' : \left(\frac{y}{a} - lp\right) \text{ et } px = F' : \left(\frac{y}{a} - lp\right)$$

patet fore:

$$f' : \left(\frac{y}{a} + lx\right) = F' : \left(\frac{y}{a} - lp\right) \text{ et}$$

$$f : \left(\frac{y}{a} + lx\right) = F : \left(\frac{y}{a} - lp\right) + F' : \left(\frac{y}{a} - lp\right)$$

vnde non ſolum relatio inter vtriusque functionis f et F indolem definitur, ſed etiam inde ſequi debet, fore:

$$px = f' : \left(\frac{y}{a} + lx\right);$$

id quod non parum videtur abſconditum. Verum ob hoc ipſum iſtud problema eo magis eſt notatu dignum, quod ſolutio altera, qua noua variabilis p introducitur, congruit cum priore, vbi z per x et y immediate definitur, neque tamen conſenſus harum ſolutionum perſpicue monſtrari poteſt. Quamobrem quando ad eiusmodi ſolutiones peruenimus, vti in problematibus poſterioribus, capitis praecedentis vſu venit, in quibus noua variabilis introducitur, non omnem ſtatim ſpem eius eliminandae abiicere debemus, cum isto casu altera solutio ad priorem certe sit reductibilis, etiamsi methodus reducendi non perspiciatur, quam tamen infra §. 119. exhibebimus.

Problema 16.

101. Si z eiusmodi esse debeat functio binarum variabilium x et y, vt posito $dz = pdx + qdy$, sit $q = pX + T$, existentibus X et T functionibus quibuscunque ipsius x, indolem istius functionis z in genere inuestigare.

Solutio.

Cum ergo sit $dz = pdx + pXdy + Tdy$, statuatur $p = r - \frac{T}{X}$ vt prodeat

$$dz = rdx - \frac{Tdx}{X} + rXdy = -\frac{Tdx}{X} + rX\left(\frac{dx}{X} + dy\right)$$

qua reductione facta perspicuum est tam rX quam

$$\int rX\left(\frac{dx}{X} + dy\right)$$

fore functionem quantitatis $y + \int\frac{dx}{X}$. Quare si ponamus:

$$\int rX\left(\frac{dx}{X} + dy\right) = f:\left(y + \int\frac{dx}{X}\right) \text{ erit}$$

$$rX = f':\left(y + \int\frac{dx}{X}\right)$$

ac tum functio quaesita erit:

$$z = -\int\frac{Tdx}{X} + f:\left(y + \int\frac{dx}{X}\right)$$

quae ob functionem indefinitam f: est completa.

Tum

Tum vero erit

$$p = \frac{-T}{X} + \frac{1}{X} f' : (y + \int \frac{dx}{X}) \text{ et}$$

$$q = f' : (y + \int \frac{dx}{X})$$

vnde patet fore vtique $q = pX + T$. Quoniam vero X et T sunt functiones datae ipsius x, formulae integrales $\int \frac{dx}{X}$ et $\int \frac{T dx}{X}$ solutionem non turbant.

Coroll. 1.

102. Solutio aliquanto facilior redditur sumendo ex conditione praescripta $p = \frac{q}{X} - \frac{T}{X}$, vnde fit

$$dz = -\frac{T dx}{X} + \frac{q dx}{X} + q dy \text{ et}$$

$$z = -\int \frac{T dx}{X} + \int q (dy + \frac{dx}{X}).$$

Iam manifesto est

$$\int q (dy + \frac{dx}{X}) = f : (y + \int \frac{dx}{X}),$$

sicque ipsa solutio praecedens resultat.

Coroll. 2.

103. Eodem modo resoluitur problema, si proponatur conditio $q = pY + V$ existentibus Y et V functionibus datis ipsius y. Tum enim erit

$dz = pdx + pY dy + V dy$ et $z = \int V dy + \int p(dx + Y dy)$.

Hic ergo fit

$$\int p(dx + Y dy) = f : (x + \int Y dy)$$

et

et ſolutio erit

$$z = \int V\,dy + f:(x + \int Y\,dy);$$

vnde fit

$$p = f':(x + \int Y\,dy) \text{ et } q = V + Yf':(x + \int Y\,dy).$$

Scholion.

104. Ex forma ſolutionis hic inuentae diſcere poterimus, quomodo problema comparatum eſſe debeat, vt eius ſolutio hac ratione perfici, et functio z per binas variabiles x et y exhiberi queat. Sint enim K et V functiones quaecunque ipſarum x et y, indeque differentiando

$$dK = L\,dx + M\,dy \text{ et } dV = P\,dx + Q\,dy$$

iam a ſolutione incipiamus, ponamusque

$$z = K + f:V$$

eritque differentiando

$$dz = L\,dx + M\,dy + (P\,dx + Q\,dy)\,f':V.$$

Cum iam hanc formam cum aſſumta

$$dz = p\,dx + q\,dy$$

comparando fit

$$p = L + Pf':V \text{ et } q = M + Qf':V, \text{ erit}$$

$$Qp - Pq = LQ - MP.$$

Quare ſi hoc problema proponatur, vt poſito

$$dz = p\,dx + q\,dy$$

fieri

fieri debeat

$$q = \frac{Q}{P} p + M - \frac{LQ}{P},$$

solutio erit $z = K + f : V$, dummodo M et L itemque P et Q ita sint comparatae vt sit

$$L dx + M dy = dK \text{ et } P dx + Q dy = dV$$

verum hi casus ad sequens caput sunt referendi.

Problema 17.

105. Si z eiusmodi esse debeat functio binarum variabilium x et y vt posito $dz = p dx + q dy$, sit $q = Px + \Pi$ existentibus P et Π functionibus datis ipsius p; indolem istius functionis z in genere investigare.

Solutio.

Cum igitur sit

$$dz = p dx + P x dy + \Pi dy \text{ erit}$$

$$z = px + \int (P x dy + \Pi dy - x dp).$$

Statuatur $Px + \Pi = v$, vt sit $x = \frac{v - \Pi}{P}$, fietque

$$z = px + \int \left(v dy - \frac{v dp}{P} + \frac{\Pi dp}{P}\right).$$

Quare cum P et Π sint functiones ipsius p, ideoque formula $\int \frac{\Pi dp}{P}$ data, habebitur

$$z = px + \int \frac{\Pi dp}{P} + \int v \left(dy - \frac{dp}{P}\right),$$

vnde patet tam v quam $\int v \left(dy - \frac{dp}{P}\right)$ functionem esse

 debere

debere formulae $y-\int\frac{dp}{P}$. Ponamus ergo

$$\int v(dy-\frac{dp}{P})=f:(y-\int\frac{dp}{P})$$

eritque

$$v=Px+\Pi=f':(y-\int\frac{dp}{P}),$$

et hinc

$$x=\frac{-\Pi}{P}+\frac{1}{P}f':(y-\int\frac{dp}{P})$$

tum vero

$$z=\int\frac{\Pi dp}{P}-\frac{\Pi p}{P}+\frac{p}{P}f':(y-\int\frac{dp}{P})+f:(y-\int\frac{dp}{P}).$$

Coroll. 1

106. In solutione huius problematis iterum noua variabilis p introducitur, ex qua cum y coniunctim primo variabilis x, tum vero ipsa functio quaesita z determinatur.

Coroll. 2.

107. Neque vero hinc istam nouam variabilem p ex calculo elidere licet; vti ante vsu venit, propterea quod hic P et Π functiones ipsius p denotant, quarum indoles iam in ipsum problema ingreditur.

Coroll. 3.

108. Simili modo problema resoluetur, si permutandis x et y, quantitas p ita per y et q detur vt sit $p=Qy+\Xi$, denotantibus Q et Ξ functiones datas ipsius q.

Scholion.

Scholion.

109. In hoc capite constituimus eiusmodi problemata tractare, quorum conditio aequatione inter binas formulas differentiales $(\frac{dz}{dx})=p$, $(\frac{dz}{dy})=q$ et vnam ex tribus variabilibus x, y et z vtcunque exprimitur. Problemata autem bina euoluta ex hoc genere certos casus complectuntur, quorum solutio peculiari methodo expediri potest simulque ad formulas simpliciores perducitur. In posteriori quidem relationem inter p, q et x ita assumsimus, vt sit $q=Px+\Pi$, seu vt in valore ipsius q per p et x expresso quantitas x vnam dimensionem non excedat; in priori vero ita vt sit $q=pX+T$, seu vt in valore ipsius q per p et x expresso, quantitas p vnicam obtineat dimensionem. In genere autem notasse iuuabit, tam quantitates p et x quam q et y inter se esse permutabiles. Cum enim sit

$$\int p\,dx = px - \int x\,dp,$$

loco

$$z = \int (p\,dx + q\,dy)$$

erit

$$z = px + \int (q\,dy - x\,dp).$$

Simili modo est

$$z = qy + \int (p\,dx - y\,dq)$$

tum vero etiam

$$z = px + qy - \int (x\,dp + y\,dq).$$

Quibus ergo casibus vna harum quatuor formularum

 inte-

integralium redditur integrabilis, iisdem ternae reliquae etiam integrationem admittent. Cum igitur in ſuperiori capite primam formulam reſoluerimus, ſi p vel q quomodocunque detur per x et y; ita eodem modo reſoluetur formula ſecunda, ſi q per p et y, tertia autem ſi p per x et q, at quarta ſi vel x per p et q vel y per p et q vtcunque datur; quae quaeſtiones cum generaliter expediri queant, eas in ſequenti problemate euoluamus.

Problema 18.

110. Poſito $dz = pdx + qdy$, ſi relatio inter p, q et x aequatione quacunque definiatur, indolem functionis z, quemadmodum ex binis variabilibus x et y determinetur in genere inueſtigare.

Solutio.

Ex aequatione inter p, q et x propoſita quaeratur valor ipſius x qui functioni cuipiam ipſarum p et q aequabitur. Cum iam ſit

$$z = px + qy - \int (xdp + ydq),$$

quoniam x eſt functio data ipſarum p et q formula xdp integretur ſumta quantitate q conſtante, ſitque

$$\int xdp = V + f:q,$$

et erit V functio cognita ipſarum p et q, qua differentiata prodeat

$$dV = xdp + Sdq,$$

vbi

vbi S quoque erit functio data ipsarum p et q. Quia iam forma $\int(xdp+ydq)$ integrationem admittere debet, aequabitur formae $V+f:q$, vnde differentiando concluditur:

$$xdp+ydq=xdp+Sdq+dqf':q$$

ideoque

$$y=S+f':q \text{ et } z=px+qy-V-f:q$$

seu $z=px+Sq+qf':q-f:q-V$

solutio ergo ita se habet: Primo ex conditione praescripta datur x per p et q; tum sumta q constante fit $V=\int xdp$, et vicissim $dV=xdp+Sdq$; inuentis autem V et S per p et q, reliquae quantitates y et z ita per easdem exprimentur vt sit

$$y=S+f':q \text{ et } z=px+Sq+qf':q-f:q-V$$

quae solutio, quia $f:q$ functionem quamcunque ipsius q siue continuam siue discontinuam denotat, vtique pro completa latissimeque patente est habenda.

Aliter.

111. Vel ex aequatione inter p, q et x data, quaeratur valor ipsius p per x et q expressus, ita vt p aequetur functioni cuipiam datae binarum variabilium x et q, per quas etiam reliquas quantitates y et z definire conemur. Ad hoc vtamur formula:

$$z=qy+\int(pdx-ydq);$$

 et

et quia p est functio ipsarum x et q dabitur earundem eiusmodi functio V vt sit

$$dV = p\,dx + R\,dq.$$

Statuatur ergo

$$\int (p\,dx - y\,dq) = V + f:q$$

eritque

$$y = -R - f':q \text{ et } z = qy + V + f:q.$$

Coroll. 1.

112. Vtraque solutio aeque commode adhiberi potest, si ex relatione inter p, q et x proposita, tam quantitatem x quam p aeque commode definire liceat. Sin autem earum altera commodius definiri queat, ea solutione, quae ad istum casum est accommodata erit vtendum.

Coroll. 2.

113. Sin autem neque p neque x commode elici queat, tum nihilo minus hic resolutio aequationum cuiusque ordinis, quin etiam transcendentium tanquam concessa assumitur. Ceterum etiamsi q facile per p et x definiatur, hinc calculus nihil iuuatur.

Coroll. 3.

114. Ex hoc problemate vtpote latissime patente etiam bina praecedentia resolui possunt; solutio autem hinc inuenta a praecedente discrepabit, cum illa

illa ex methodo particulari fit deducta operae autem pretium erit has duplices solutiones inter se comparare.

Exemplum 1.

115. *Si fuerit* $q = pX + T$ *existentibus* X *et* T *functionibus ipsius* x, *indolem functionis* z *inuestigare.*

Hic solutione vtendum est posteriori, pro qua est $p = \frac{q-T}{X}$; nunc posita q constante prodit

$$V = \int p\,dx = q\int \frac{dx}{X} - \int \frac{T\,dx}{X},$$

hincque

$$R = \left(\frac{dV}{dq}\right) = \int \frac{dx}{X};$$

vnde solutio his formulis continetur

$$q = pX + T;\ y = -\int \frac{dx}{X} - f' : q;\ z = -\int \frac{T\,dx}{X} - qf' : q + f : q$$

solutio autem superior ita se habebat:

$$q = pX + T;\ q = f' : \left(y + \int \frac{dx}{X}\right) \text{ et } z = -\int \frac{T\,dx}{X} + f : \left(y + \int \frac{dx}{X}\right)$$

Scholion.

116. Consensus harum duarum solutionum ita ostendi potest vt ex ea, quam hic inuenimus, antecedens per legitimam consequentiam formetur. Cum enim sit

$$f' : q = -y - \int \frac{dx}{X}$$

statuatur breuitatis gratia $y + \int \frac{dx}{X} = v$, vt sit $f' : q = -v$; erit ergo vicissim q aequalis functioni cuidam ipsius v,

quae

quae ponatur $q = F':v$, vnde fit $dq = dv F'':v$, ergo

$$dq f':q = -v dv F'':v = -v d.F':v,$$

ergo integrando

$$f:q = -\int v d.F':v = -vF':v + \int dv.F':v = -vF':v + F:v.$$

Quare cum sit

$$z = -\int \frac{T dx}{X} - q f':q + f:q \text{ erit}$$

$$z = -\int \frac{T dx}{X} + vF':v - vF':v + F:v \text{ seu}$$

$$z = -\int \frac{T dx}{X} + F:(y + \int \frac{dx}{X})$$

quae est ipsa solutio praecedens.

Exemplum 2.

117. *Si fuerit* $q = Px + \Pi$ *existentibus* P *et* Π *functionibus datis ipsius* p, *indolem functionis* z, *vt sit* $dz = pdx + qdy$ *inuestigare.*

Hic solutione priori vtendum, cum sit $x = \frac{q - \Pi}{P}$. Sumto ergo q constante quaeratur

$$V = \int x dp = q \int \frac{dp}{P} - \int \frac{\Pi dp}{P},$$

vnde fit

$$S = \left(\frac{dV}{dq}\right) = \int \frac{dp}{P}.$$

Solutio ergo praebet:

$$y = \int \frac{dp}{P} + f':q \text{ et}$$

$$z = \frac{pq}{P} - \frac{p\Pi}{P} + q\int \frac{dp}{P} + qf':q - f:q - q\int \frac{dp}{P} + \int \frac{\Pi dp}{P} \text{ siue}$$

$$z = \frac{p(q - \Pi)}{P} + \int \frac{\Pi dp}{P} + qf':q - f:q$$

solutio

Solutio autem eiusdem casus supra (105.) inuenta erat:

$$x = \frac{-\Pi}{P} + \frac{1}{P} f' : (y - \int \frac{dp}{P}) \text{ et } q = Px + \Pi \text{ atque}$$

$$z = \frac{-p\Pi}{P} + \int \frac{\Pi dp}{P} + \frac{p}{P} f' : (y - \int \frac{dp}{P}) + f : (y - \int \frac{dp}{P}).$$

Scholion 1.

118. Videamus quomodo solutio hic inuenta ad superiorem reduci queat. Cum ibi inuenerimus

$$y - \int \frac{dp}{P} = f' : q,$$

vicissim q aequabitur functioni quantitatis $y - \int \frac{dp}{P}$, ponatur ergo

$$q = F' : (y - \int \frac{dp}{P})$$

eritque statim

$$x = \frac{-\Pi}{P} + \frac{1}{P} F' : (y - \int \frac{dp}{P});$$

sit breuitatis gratia $y - \int \frac{dp}{P} = v$, vt fiat

$$q = F' : v \quad \text{et} \quad v = f' : q, \text{ erit}$$

$$F : v = \int q dv = qv - \int v dq = qv - \int dq f' : q.$$

Ergo $F : v = qv - f : q$, ita vt sit

$$f : q = q(y - \int \frac{dp}{P}) - F : (y - \int \frac{dp}{P}) \text{ seu}$$

$$f : q = (y - \int \frac{dp}{P}) F' : (y - \int \frac{dp}{P}) - F : (y - \int \frac{dp}{P}).$$

 Quibus

Quibus valoribus ſubſtitutis habebimus:

$$x = \frac{-\pi}{p} + \frac{1}{p} F' : \left(y - \int \frac{dp}{P}\right) \text{ et}$$

$$z = \frac{-p\pi}{p} + \frac{p}{p} F' : \left(y - \int \frac{dp}{P}\right) + \int \frac{\pi\, dp}{P} + \left(y - \int \frac{dp}{P}\right) F' : \left(y - \int \frac{dp}{P}\right)$$
$$- \left(y - \int \frac{dp}{P}\right) F' : \left(y - \int \frac{dp}{P}\right) + F : \left(y - \int \frac{dp}{P}\right)$$

ſeu $z = \frac{-p\pi}{P} + \frac{p}{P} F' : \left(y - \int \frac{dp}{P}\right) + \int \frac{\pi\, dp}{P} + F : \left(y - \int \frac{dp}{P}\right)$

quae eſt ipſa ſolutio ante inuenta.

Scholion 2.

119. Hoc conſenſu oſtenſo etiam conſenſum ſupra obſeruatum (100.) demonſtrare poterimus, qui multo magis abſconditus videtur. Altera autem ſolutio ibi inuenta erat

$$px = F' : \left(\frac{y}{a} - lp\right) \text{ et } z = px + F : \left(\frac{y}{a} - lp\right),$$

ex quarum formula priori patet fore viciſſim $\frac{y}{a} - lp$ functionem ipſius px; hinc etiam $\frac{y}{a} - lp + lpx$ ſeu $\frac{y}{a} + lx$ aequabitur functioni ipſius px. Denuo ergo viciſſim px aequabitur functioni cuipiam ipſius $\frac{y}{a} + lx$. Ponatur ergo $px = f' : \left(\frac{y}{a} + lx\right)$; et cum ſit

$$d.F : \left(\frac{y}{a} - lp\right) = \left(\frac{dy}{a} - \frac{dp}{p}\right) F' : \left(\frac{y}{a} - lp\right), \text{ erit}$$

$$F : \left(\frac{y}{a} - lp\right) = \int px \left(\frac{dy}{a} - \frac{dp}{p}\right) = \int px \left(\frac{dy}{a} + \frac{dx}{x}\right) - \int px \left(\frac{dx}{x} + \frac{dp}{p}\right)$$
$$= \int px \left(\frac{dy}{a} + \frac{dx}{x}\right) - px.$$

Iam pro px ſubſtituto valore $f' : \left(\frac{y}{a} + lx\right)$ obtinebitur:

$$F : \left(\frac{y}{a} - lp\right) = -px + \int \left(\frac{dy}{a} + \frac{dx}{x}\right) f' : \left(\frac{y}{a} + lx\right) = -px + f : \left(\frac{y}{a} + lx\right)$$

ita

ita vt hinc fiat $z = f:(\frac{2}{a} + lx)$, quae eſt ipſa ſolutio altera. Hac igitur reductione haud parum luminis accenditur ad alia myſteria huius generis inveſtiganda. Summa autem huius ratiocinii huc redit, vt ſi fuerit $r = f':s$ fore etiam $r = F':(s+R)$ denotante R functionem ipſius r, quod quidem per ſe eſt euidens, quia vtrinque r per s determinatur. Cum ergo ſit

$$f':s = r = F':(s+R), \text{ erit}$$

$$f:s = \int ds f':s = \int r\,ds = \int r(ds + dR - dR) = \int (ds + dR)F':(s+R) - \int r\,dR,$$

ideoque $f:s = F:(s+R) - \int r\,dR$;

vnde loco functionum quantitatis s, functiones quantitatis $s+R$ introduci poſſunt. Scilicet ſi ſit $r = f':s$ ſumi poteſt $r = F':(s+R)$ exiſtente R functione quacunque ipſius r, tum vero erit

$$f:s = F:(s+R) - \int r\,dR.$$

Exemplum 3.

120. *Poſito* $dz = p\,dx + q\,dy$, *ſi* x *aequetur functioni homogeneae* n *dimenſionum ipſarum* p *et* q, *indolem functionis* z *inueſtigare.*

Cum x detur per p et q, vtendum erit ſolutione priori et ob $x =$ functioni homogeneae n dimenſionum ipſarum p et q, ponatur $p = qr$, fietque $x = q^n R$, exiſtente R functione ipſius r tantum. Sumatur nunc q conſtans et quaeratur

$$V = \int x\,dp = \int q^{n+1} R\,dr$$

ob $dp = qdr$, eritque

$$V = q^{n+1}\int R\,dr,$$

quod integrale datur. Hinc differentiando erit

$$dV = q^{n+1}R\,dr + (n+1)q^n\,dq\int R\,dr$$

quae vt cum

$$dV = x\,dp + S\,dq = q^n R\,dp + S\,dq$$

comparari possit, quia ob $dp = q\,dr + r\,dq$ est

$$dV = q^{n+1}R\,dr + q^n Rr\,dq + S\,dq \text{ erit}$$

$$S = -q^n Rr + (n+1)q^n\int R\,dr,$$

vnde fit

$y = -q^n Rr + (n+1)q^n\int R\,dr + f':q$ et $x = q^n R$ atque
$z = nq^{n+1}\int R\,dr + qf':q - f:q$ existente $p = qr$.

Coroll. 1.

121. Sit $x = \frac{p^m}{q^m}$, et posito $p = qr$ erit $x = r^m$, ideoque $n = 0$ et $R = r^m$; vnde fit

$y = -r^{m+1} + \frac{r^{m+1}}{m+1} + f':q = \frac{-m}{m+1}r^{m+1} + f':q$ et $z = qf':q - f:q$.

Quare ob $r = x^{\frac{1}{m}}$ erit $y = \frac{-m}{m+1}x^{\frac{m+1}{m}} + f':q$.

Coroll. 2.

122. Eodem casu ergo quo $x = \frac{p^m}{q^m}$, aequa-

bitur:

bitur q functioni quantitatis $y+\frac{m}{m+1}x^{\frac{m+1}{m}}$, quae quantitas si ponatur $=v$ et $q=F':v$, vt sit $v=f':q$, crit

$$f:q=\int dqf':q=\int vdvF'':v \text{ ob } dq=dvF'':v,$$

vnde concluditur

$$f:q=vF':v-F:v \text{ et } z=F:v=F:\left(y+\frac{m}{m+1}x^{\frac{m+1}{m}}\right).$$

Exemplum 4.

123. Duarum variabilium x et y eiusmodi functionem z inuestigare vt posito

$$dz=pdx+qdy \text{ fiat } p^3+x^3=3pqx.$$

Solutio.

Consideretur forma

$$z=qy+\int(pdx-ydq),$$

vbi iam formulam $pdx-ydq$ integrabilem reddi oportet. Statuatur $p=ux$, et conditio praescripta dat

$$x(1+u^3)=3qu;$$

vnde fit

$$x=\frac{3qu}{1+u^3} \text{ et } p=\frac{3quu}{1+u^3}$$

tum vero

$$dx=\frac{3qdu(1-2u^3)}{(1+u^3)^2}+\frac{3udq}{1+u^3},$$

sicque habebitur

$$z=qy+\int\left(\frac{9qquudu(1-2u^3)}{(1+u^3)^3}+\frac{9qu^3dq}{(1+u^3)^2}-ydq\right)$$

at $\int\frac{9qquudu(1-2u^3)}{(1+u^3)^3}=\frac{3qq(1+4u^3)}{2(1+u^3)^2}-\int\frac{3q(1+4u^3)dq}{(1+u^3)^2}$.

 Ergo

Ergo $z = qy + \frac{3qq(1+4u^3)}{2(1+u^3)^2} - \int dq\left(y + \frac{3q}{1+u^3}\right)$.
Quare necesse est esse $y + \frac{3q}{1+u^3}$ functionem ipsius q tantum, quae sit $= -f':q$ vnde fit

$$y = -\frac{3q}{1+u^3} - f':q \text{ et } z = qy + \frac{3qq(1+4u^3)}{2(1+u^3)^2} + f:q$$

seu $z = \frac{3qq(2u^3-1)}{2(1+u^3)^2} - qf':q + f:q$ existente $x = \frac{3qu}{1+u^3}$.
Ex quibus tribus aequationibus si elimentur binae quantitates q et u orietur aequatio inter z et x, y, quae quaeritur.

Coroll. 1.

124. Ex aequatione pro y inuenta colligitur $\frac{3}{1+u^3} = \frac{-y-f':q}{q}$ aequatio autem pro z inuenta abit in hanc:

$$z = \frac{3qq}{1+u^3} - \frac{9qq}{2(1+u^3)^2} - qf':q + f:q$$

quae eliso u transmutatur in hanc

$$z = -qy - 2qf':q - \tfrac{1}{2}(y+f':q)^2 + f:q;$$

tum vero est

$$x = -u(y+f':q)$$

vnde reperitur $u = \frac{-x}{y+f':q}$, hincque

$$x^3 = 3q(y+f':q)^2 + (y+f':q)^3.$$

Coroll. 2.

125. Si sumamus $f':q = a$, erit $f:q = aq+b$, et postrema aequatio praebet $q = \frac{x^3-(y+a)^3}{3(y+a)^2}$. Cum deinde pro hoc casu fiat

$$z = -qy - aq - \tfrac{1}{2}(y+a)^2 + b$$

proue-

proueniet loco q valorem inuentum substituendo

$$z = \frac{6b(y+a) - (y+a)^3 - 2x^3}{6(y+a)}.$$

Coroll. 3.

126. Cum in genere sit

$$x^3 = (y + f':q)^2 (y + 3q + f':q)$$

ponamus $f':q = a - 3q$, ideoque $f:q = b + aq - \frac{3}{2}qq$, vt fiat $(y + a - 3q)^2 = \frac{x^3}{y+a}$ eritque

$$y + a - 3q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} \text{ et } q = \tfrac{1}{3}(y+a) - \frac{x\sqrt{x}}{3\sqrt{(y+a)}}.$$

Hinc ergo prodit

$$f':q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} - y \text{ et}$$

$$f:q = b + \frac{a(y+a)}{3} - \frac{ax\sqrt{x}}{3\sqrt{(y+a)}} - \tfrac{1}{6}(y+a)^2 + \tfrac{1}{3}x\sqrt{x(y+a)}$$
$$- \frac{x^3}{6(y+a)} = b + \frac{aa - yy}{6} + \frac{xy\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{6(y+a)}$$

atque

$$z = -\tfrac{1}{3}y(y+a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - 2aq + 6qq - \frac{x^3}{2(y+a)} + b + aq - \tfrac{3}{2}qq$$

seu $z = b - \frac{1}{3}y(y+a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{2(y+a)} - aq + \frac{9}{2}qq$

et facta reductione

$$z = b + \tfrac{1}{6}(y+a)^2 - \tfrac{2}{3}x\sqrt{x(y+a)};$$

Coroll. 4.

127. Quodsi hic sumatur $a = 0$ et $b = 0$ erit per expressionem satis simplicem

$$z = \tfrac{1}{6}yy - \tfrac{2}{3}x\sqrt{xy};$$

quae quomodo conditioni praescriptae satisfaciat, ita apparet. Per differentiationem colligitur

$$p = \left(\frac{dz}{dx}\right) = -\sqrt{xy} \text{ et } q = \left(\frac{dz}{dy}\right) = \tfrac{1}{3}y - \frac{x\sqrt{x}}{3\sqrt{y}}$$

hinc-

hincque:

$$p^3 + x^3 = -xy\sqrt{xy} + x^3, \text{ at}$$

$$3pq = xx - y\sqrt{xy} \text{ ideoque}$$

$$3pqx = x^3 - xy\sqrt{xy} \text{ ergo}$$

$$p^3 + x^3 = 3pqx.$$

Scholion.

128. Succeſſit ergo ſolutio, quando aequatio quaecunque inter p, q et x proponitur, etiamſi caſibus, quibus inde neque x neque p elici poteſt, difficultas quaedam reſtat, quae autem reſolutionem aequationum finitarum potiſſimum afficit, quam hic merito concedi poſtulamus. Interim ex poſtremo exemplo perſpicitur, quomodo operatio ſit inſtituenda, ſi ope ſubſtitutionis idoneae aequatio propoſita ad reſolutionem accommodari queat, cui autem negotio hic amplius non immoror. Neque etiam eos caſus, quibus inter p, q et y relatio quaedam praeſcribitur, hic ſeorſim euoluam, cum ob permutabilitatem ipſarum x et y, qua etiam p et q permutantur, hi caſus ad praecedentes ſponte reuocentur. Supereſt igitur caſus, quo aequatio inter p, q et z proponitur, vbi quidem ſtatim manifeſtum eſt, in aequatione $dz = pdx + qdy$, quantitates p et q non vti functiones ipſarum x et y ſpectari poſſe, quoniam etiam a z pendent, neque ergo earum indoles inde determinari poterit, vt formula $pdx + qdy$ integrabilis euadat. Verum ſine diſcrimine conditio

ea

ea est definienda, vt aequatio differentialis

$$dz - p\,dx - q\,dy = 0$$

fiat possibilis; ad quod ex principiis supra stabilitis (6) requiritur vt posito

$$\left(\frac{dq}{dz}\right) = L;\ -\left(\frac{dp}{dz}\right) = M \text{ et } \left(\frac{dp}{dy}\right) - \left(\frac{dq}{dx}\right) = N \text{ fit}$$

$$Lp + Mq - N = 0 \text{ seu } p\left(\frac{dq}{dz}\right) - q\left(\frac{dp}{dz}\right) + \left(\frac{dq}{dx}\right) - \left(\frac{dp}{dy}\right) = 0.$$

Quare proposita aequatione quacunque inter p, q et z eas conditiones in genere inuestigari oportet, vt huic requisito satisfiat.

Problema 19.

129. Si posito $dz = p\,dx + q\,dy$, debeat esse $p + q = \frac{z}{a}$, relationem functionis z ad variabiles x et y in genere inuestigare.

Solutio.

Cum sit $q = \frac{z}{a} - p$, aequatio nostra hanc induet formam

$$dz = p\,dx - p\,dy + \frac{z\,dy}{a} \text{ seu}$$

$$p(dx - dy) = \frac{a\,dz - z\,dy}{a} = z\left(\frac{dz}{z} - \frac{dy}{a}\right).$$

Quoniam igitur ambae formulae

$$dx - dy \text{ et } \frac{dz}{z} - \frac{dy}{a}$$

per se sunt integrabiles, ob

$$\frac{dz}{z} - \frac{dy}{a} = \frac{p}{z}(dx - dy),$$

 necesse

necesse est vt $\frac{p}{z}$ sit functio quantitatis $x-y$, ponatur ergo

$$\frac{p}{z}=f':(x-y) \text{ vt fiat } lz-\frac{y}{a}=f:(x-y).$$

Definiri ergo potest z per x et y, et cum sit $e^{f:(x-y)}$ etiam functio ipsius $x-y$, si ea ponatur $=F:(x-y)$ erit

$$z=e^{\frac{y}{a}}F:(x-y), \text{ vnde fit}$$

$$\left(\frac{dz}{dx}\right)=p=e^{\frac{y}{a}}F':(x-y) \text{ et}$$

$$\left(\frac{dz}{dy}\right)=q=-e^{\frac{y}{a}}F':(x-y)+\frac{1}{a}e^{\frac{y}{a}}F:(x-y)$$

ideoque

$$p+q=\frac{1}{a}e^{\frac{y}{a}}F:(x-y)=\frac{z}{a},$$

vti requiritur.

Coroll. 1.

130. Ex hoc exemplo intelligitur, quomodo certa functio ipsarum p et q, quantitati z aequari possit, etiamsi p et q sint functiones ipsarum x et y. Simul scilicet ratio integralis formulae

$$dz=p\,dx+q\,dy$$

introducitur in calculum.

Coroll. 2.

Coroll. 2.

131. Forma $e^{\frac{y}{a}}$ F$:(x-y)$ pro valore ipſius z inuenta per functionem quamuis ipſius $x-y$ multiplicari poteſt. Si ergo multiplicetur per

$$e^{\frac{x-y}{a}} \text{ fit } z = e^{\frac{x}{a}} \mathrm{F}:(x-y).$$

Sin autem multiplicetur per

$$e^{\frac{x-y}{2a}} \text{ fit } z = e^{\frac{x+y}{2a}} \mathrm{F}:(x-y),$$

quae formae problemati aeque ſatisfaciunt.

Problema 20.

132. Si poſito $dz = pdx + qdy$, quantitas z aequari debeat functioni datae ipſarum p et q, indolem, qua z per x et y definitur, in genere inveſtigare.

Solutio.

Ex formula propoſita habemus $dy = \frac{dz}{q} - \frac{pdx}{q}$; ſtatuatur $p = qr$, vt ſit z aequalis functioni ipſarum q et r, et ex $dy = \frac{dz}{q} - rdx$ elicitur

$$y = \frac{z}{q} - rx + \int \left(\frac{zdq}{qq} + xdr\right),$$

quam formulam integrabilem reddi oportet. Cum igitur z ſit functio data ipſarum q et r, poſito r conſtante quaeratur integrale formulae $\frac{zdq}{qq}$ ſitque

$$\int \frac{zdq}{qq} = \mathrm{V} + f:r$$

 vnde

vnde differentiando prodeat

$$dV = \frac{z\,dq}{qq} + R\,dr,$$

ac iam patet esse debere $x = R + f':r$, indeque obtineri

$$y = \frac{z}{q} - Rr - rf':r + V + f:r,$$

quibus duabus aequationibus relatio inter quantitates propositas determinatur. Primo igitur posito $p = qr$ datur z per q et r. Deinde sumto r constante integretur formula $\frac{z\,dq}{qq}$, sitque integrale resultans $V = \int \frac{z\,dq}{qq}$, quod etiam per q et r datur; vnde sumto q constante colligitur $R = \left(\frac{dV}{dr}\right)$. Quibus inventis erit

$$x = R + f':r \text{ et } y = \frac{z}{q} - rx + V + f:r,$$

sicque omnes quantitates per binas variabiles q et r determinantur.

Coroll. 1.

133. Quia permutatis x et y litterae p et q permutantur, simili modo nostram inuestigationem incipere potuissemus ab aequatione

$$dx = \frac{dz}{p} - \frac{q\,dy}{p};$$

similisque solutio prodiisset quae quidem forma diversa at re congruens esset.

Coroll. 2.

Coroll. 2.

134. Iam ſcilicet poſito $q = ps$, vt ſit
$dx = \frac{dz}{p} - s\,dy$, erit

$$x = \frac{z}{p} - sy + \int\left(\frac{z\,dp}{pp} + y\,ds\right).$$

Iam ſumto s conſtante ponatur $\int \frac{z\,dp}{pp} = \mathrm{U}$, quae quantitas per p et s determinatur, ex ea vero prodeat $\left(\frac{d\mathrm{U}}{ds}\right) = \mathrm{S}$, erit

$$y = \mathrm{S} + f' : s \text{ et } x = \frac{z}{p} - sy + \mathrm{U} + f : s.$$

Exemplum 1.

135. *Si eſſe debeat* $\mathrm{p} + \mathrm{q} = \frac{z}{a}$, *ſolutionem pro hoc caſu exhibere.*

Poſito $p = qr$ erit $z = aq(1+r)$, nunc ſumto r conſtante erit:

$$\mathrm{V} = \int \frac{z\,dq}{qq} = a(1+r)lq \text{ et } \mathrm{R} = \left(\frac{d\mathrm{V}}{dr}\right) = alq.$$

Hinc reperitur

$$x = alq + f' : r \text{ et } y = \frac{z}{q} - arlq - rf' : r + a(1+r)lq + f : r$$

ſeu $y = a(1+r) + alq - rf' : r + f : r$.

Si hinc q elidere velimus ob $q = \frac{z}{a(1+r)}$, ſolutio his duabus aequationibus continetur

$$x = al\frac{z}{a(1+r)} + f' : r \text{ et}$$

$$y = al\frac{z}{a(1+r)} + a(1+r) - rf' : r + f : r.$$

 Vnde

Vnde sequenti modo praecedens solutio elici potest, ex forma priori est

$$\frac{z}{a} - l\frac{z}{a} = -l(1+r) + \frac{1}{a}f':r = \text{funct.}\, r$$

ex ambabus vero

$$y - x = a(1+r) - (1+r)f':r + f:r = \text{funct.}\, r.$$

Cum ergo tam $\frac{z}{a} - l\frac{z}{a}$ seu $ze^{-\frac{z}{a}}$ quam $y-x$ sit functio ipsius r altera forma aequabitur functioni alterius,, vnde statui potest

$$ze^{-\frac{z}{a}} = \text{F}:(y-x) \text{ seu } z = e^{\frac{z}{a}}\text{F}:(y-x),$$

quae est solutio ante inuenta.

Exemplum 2.

136. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = apq$, *relationem inter* x, y *et* z *inuestigare.*

Posito $p = qr$ erit $z = aqqr$, et sumto r constante fit $\text{V} = \int\frac{z\,dq}{qq} = aqr$, hincque $\text{R} = (\frac{d\text{V}}{dr}) = aq$. Quocirca habebimus

$$x = aq + f':r \text{ et } y = aqr - rf':r + f:r$$

seu ob $r = \frac{z}{aqq}$ erit

$$x = aq + f':\frac{z}{aqq} \text{ et } y = \frac{z}{q} - \frac{z}{aqq}f':\frac{z}{aqq} + f:\frac{z}{aqq}.$$

Hic in genere notemus si sit $f':r = v$, ponamusque $r = \text{F}':v$ ob $dr = dv\text{F}'':v$ fore

$$f:r = \int dr f':r = \int v dv \text{F}'':v = v\text{F}':v - \text{F}v$$

seu $f:r = v\text{F}':v - \text{F}:v$,

hincque $f:r - rf':r = -\text{F}:v$.

Quare

Quare cum sit $f':r = x - aq$, si ponamus $r = F':(x-aq)$, erit

$$f:r - rf':r = -F:(x-aq) \text{ et}$$

$$y = aqF':(x-aq) - F:(x-aq) \text{ atque}$$

$$z = aqqF':(x-aq).$$

Scholion.

137. Hae postremae formulae ita statim ex conditione quaestionis elici possunt. Nam ob $p = \frac{z}{aq}$ erit

$$dz = \frac{z\,dx}{aq} + q\,dy, \quad \text{et} \quad dy = \frac{dz}{q} - \frac{z\,dx}{aqq},$$

hincque

$$y = \frac{z}{q} + \int\left(\frac{z\,dq}{qq} - \frac{z\,dx}{aqq}\right) = \frac{z}{q} + \int\frac{z}{qq}\left(dq - \frac{dx}{a}\right)$$

vbi manifestum est esse $\frac{z}{qq}$ functionem quantitatis $q - \frac{x}{a}$. Quare posito

$$\frac{z}{qq} = F':\left(q - \frac{x}{a}\right), \text{ erit}$$

$$y = \frac{z}{q} + F:\left(q - \frac{x}{a}\right).$$

Quin etiam indidem alia solutio deduci potest ponendo

$$dx = \frac{aq}{z}(dz - q\,dy),$$

quae posito $z = qv$ abit in

$$dx = \frac{a}{v}(v\,dq + q\,dv - q\,dy), \text{ vnde}$$

$$x = aq + \int\frac{aq}{v}(dv - dy).$$

Quare ponatur

$$\frac{aq}{v} = f':(v-y) \text{ eritque } x = aq + f:(v-y).$$

Iam

Iam reſtituto valore $v=\frac{z}{q}$ habebitur:

$$\frac{aqq}{z}=f':\left(\frac{z}{q}-y\right) \text{ et } x-aq=f:\left(\frac{z}{q}-y\right).$$

Prima autem ſolutio ad eliminanda q et r eſt aptiſſima in exemplis: Si enim ponatur

$$f':r=\frac{b}{\sqrt{r}}+c \text{ erit } f:r=2b\sqrt{r}+cr+d;$$

hinc:

$$z=aqqr \text{ et } x=aq+\frac{b}{\sqrt{r}}+c;\; y=aqr+b\sqrt{r}+d.$$

Iam ob $r=\frac{z}{aqq}$ fit

$$x=aq+bq\sqrt{\frac{a}{z}}+c \text{ et } y=\frac{z}{q}+\frac{b}{q}\sqrt{\frac{z}{a}}+d.$$

Hinc

$$x-c=q\left(a+\frac{b\sqrt{a}}{\sqrt{z}}\right) \text{ et } y-d=\frac{z}{aq}\left(a+\frac{b\sqrt{a}}{\sqrt{z}}\right)$$

et multiplicando eliditur q fitque

$$(x-c)(y-d)=\frac{z}{a}\left(a+\frac{b\sqrt{a}}{\sqrt{z}}\right)^2=(b+\sqrt{az})^2$$

ita vt ſit

$$b+\sqrt{az}=\sqrt{(x-c)(y-d)}$$

et proinde

$$z=\frac{(x-c)(y-d)-2b\sqrt{(x-c)(y-d)}+bb}{a}$$

quae ſi $b=c=d=0$ dat caſum ſimpliciſſimum $z=\frac{xy}{a}$.

CAPVT V.

CAPVT V.

DE RESOLVTIONE AEQVATIONVM QVIBVS RELATIO INTER QVANTITATES $(\frac{dz}{dx})$, $(\frac{dz}{dy})$, ET BINAS TRIVM VARIABILIVM x, y, z QVAECVNQVE DATVR.

Problema 21.

138.

Si posito $dz = pdx + qdy$, debeat esse $px + qy = 0$ functionis z indolem per x et y in genere investigare.

Solutio.

Cum sit $q = -\frac{px}{y}$ erit

$$dz = pdx - \frac{pxdy}{y} = px(\frac{dx}{x} - \frac{dy}{y}) \text{ seu}$$

$$dz = py(\frac{dx}{y} - \frac{xdy}{yy}) = pyd.\frac{x}{y}.$$

Vnde patet py esse debere functionem ipsius $\frac{x}{y}$; ac si ponatur $py = f':\frac{x}{y}$ fore $z = f:\frac{x}{y}$. Perpetuo scilicet

cet in designandis functionibus hac lege vtemur, vt sit

$$d.f:v = dvf':v$$

sicque porro

$$d.f':v = dvf'':v \text{ et } d.f'':v = dvf''':v \text{ etc.}$$

At $f:\frac{x}{y}$ denotat functionem quamcunque homogeneam ipsarum x et y nullius dimensionis, ac si z fuerit talis functio quaecunque, et differentiando prodeat $dz = pdx + qdy$, semper erit $px + qy = 0$.

Coroll. 1.

139. Quodsi ergo z fuerit functio homogenea nullius dimensionis ipsarum x et y, ob

$$p = \left(\frac{dz}{dx}\right) \text{ et } q = \left(\frac{dz}{dy}\right) \text{ erit}$$

$$x\left(\frac{dz}{dx}\right) + y\left(\frac{dz}{dy}\right) = 0,$$

quam veritatem quidem iam supra elicuimus.

Coroll. 2.

140. Tum vero cum sit

$$p = \frac{1}{y} f':\frac{x}{y} \text{ et } q = \frac{-x}{yy} f':\frac{x}{y},$$

erit p functio homogenea ipsarum x et y numeri dimensionum $= -1$, et si sit $q = \frac{-px}{y}$, ipsa functio z reperitur ex integratione $z = \int pyd.\frac{x}{y}$.

Scholion.

Scholion.

141. Simili modo soluitur problema, si posito $dz=pdx+qdy$, fieri debeat $mpx+nqy=a$. Tum enim ob $q=\frac{a}{ny}-\frac{mpx}{ny}$, erit

$$dz=\frac{ady}{ny}+pdx-\frac{mpxdy}{ny} \text{ seu}$$

$$dz=\frac{ady}{ny}+\frac{px}{n}\left(\frac{ndx}{x}-\frac{mdy}{y}\right)=\frac{ady}{ny}+\frac{py^m}{nx^{n-1}}\,d.\frac{x^n}{y^m};$$

vnde solutio praebet

$$\frac{py^m}{nx^{n-1}}=f':\frac{x^n}{y^m} \text{ et } z=\frac{a}{n}ly+f:\frac{x^n}{y^m}.$$

Quin etiam hoc generalius problema resolui potest quo esse debet $pX+qY=A$; existente X functione ipsius x et Y ipsius y: Cum enim inde fiat $q=\frac{A}{Y}-\frac{pX}{Y}$, erit

$$dz=\frac{Ady}{Y}+pdx-\frac{pXdy}{Y}=\frac{Ady}{Y}+pX\left(\frac{dx}{X}-\frac{dy}{Y}\right).$$

Statui ergo debet

$$pX=f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right),$$

indeque fit

$$z=A\int\frac{dy}{Y}+f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right).$$

Problema 22.

142. Si posito $dz=pdx+qdy$ debeat esse $\frac{q}{p}$, aequale functioni datae cuicunque ipsarum x et y, indolem functionis z in genere inuestigare.

Solutio.

Sit V ista functio data ipsarum x et y, vt sit $q = pV$ et habebitur $dz = p(dx + Vdy)$. Dabitur iam multiplicator M itidem functio ipsarum x et y, vt $M(dx + Vdy)$ fiat integrabile. Ponatur ergo $M(dx + Vdy) = dS$, ac dabitur etiam S functio ipsarum x et y. Cum ergo sit $dz = \frac{p\,dS}{M}$ perspicuum est, quantitatem $\frac{p}{M}$ aequari debere functioni ipsius S, quare si ponamus $\frac{p}{M} = f':S$ fiet $z = f:S$, indeque erit

$$p = Mf':S \text{ et } q = MVf':S.$$

Coroll. 1.

143. Hoc ergo casu functio quaesita z statim inuenitur per x et y expressa, quoniam S per x et y datur. Fieri autem potest, vt S prodeat quantitas transcendens; quin etiam vt per methodos adhuc cognitas multiplicator M ne inueniri quidem possit.

Coroll. 2.

144. Si V sit functio nullius dimensionis ipsarum x et y, erit $M = \frac{1}{x + Vy}$. Seu posito $x = vy$, fiet V functio ipsius v, et

$$dS = M(y\,dv + v\,dy + V\,dy).$$

Capiatur $M = \frac{1}{y(v+V)}$, eritque

$$dS = \frac{dy}{y} + \frac{dv}{v+V}; \text{ vnde reperitur}$$

$$z = f:\left(ly + \int \frac{dv}{v+V}\right).$$

Scholion.

Scholion.

145. Ob permutabilitatem ipſarum p et x item q et y, ſimili modo ſequentia problemata reſolui poſſunt:

I. Si debeat eſſe $q = xV$, exiſtente V functione quacunque ipſarum p et y, conſideretur forma

$$z = px + \int(qdy - xdp) = px + \int x(Vdy - dp).$$

Quaeratur multiplicator M vt ſit

$$M(Vdy - dp) = dS$$

erit S functio ipſarum p et y atque

$$z = px + \int\frac{xdS}{M};$$

ex quo colligitur haec ſolutio

$$\frac{x}{M} = f':S \text{ et } z = pMf':S + f:S.$$

II. Si debeat eſſe $y = pV$, exiſtente V functione quacunque ipſarum x et q. Conſideretur forma

$$z = qy + \int(pdx - ydq) = qy + \int p(dx - Vdq).$$

Quaeratur multiplicator M vt ſit

$$M(dx - Vdq) = dS$$

erit S functio ipſarum x et q et

$$z = qy + \int\frac{pdS}{M}.$$

Quare fit

$$\frac{p}{M} = f':S \text{ et } z = qy + f:S$$

 ſeu

seu ob $p = \frac{z}{V}$ erit

$$y = MVf':S \quad \text{et} \quad z = qMVf':S + f:S.$$

III. Si debeat esse $y = xV$ existente V functione quacunque ipsarum p et q, consideretur haec forma:

$$z = px + qy - \int (x\,dp + xV\,dq).$$

Quaeratur multiplicator M vt fiat

$$M(dp + V\,dq) = dS$$

erit S functio ipsarum p et q et

$$z = px + qy - \int \frac{x\,dS}{M},$$

vnde haec solutio nascitur

$$\frac{x}{M} = f':S \quad \text{et} \quad z = px + qy - f:S.$$

Omnes hi casus huc redeunt, vt quaternarum quantitatum p, x, q, y, vel $\frac{q}{p}$, vel $\frac{q}{x}$ vel $\frac{y}{p}$ vel $\frac{y}{x}$ aequetur functioni cuicunque binarum reliquarum.

Problema 23.

146. Si posito $dz = p\,dx + q\,dy$, requiratur vt sit $q = pV + U$, existente tam V quam U functione quacunque binarum variabilium x et y, indolem functionis z in genere inuestigare.

Solutio.

Cum ob $q = pV + U$ fit

$$dz = p(dx + V\,dy) + U\,dy$$

quae-

quaeratur primo multiplicator M formulam $dx+Vdy$ reddens integrabilem, sitque

$$M(dx+Vdy)=dS,$$

erunt M et S functiones ipsarum x et y, fietque

$$dz=\frac{p\,dS}{M}+U\,dy.$$

Cum iam sit S functio ipsarum x et y inde x per y et S definiri potest quo valore introducto fient U et M functiones ipsarum y et S. Nunc sumto S constante, integretur formula $U\,dy$ sitque

$$\int U\,dy=T+f:S,$$

ac posito

$$dT=U\,dy+W\,dS, \text{ fiet}$$

$$\frac{p}{M}=W+f':S \text{ et } z=T+f:S,$$

sicque omnia per binas variabiles y et S exprimentur.

Coroll. 1.

147. Datis ergo binarum variabilium x et y functionibus V et U vt sit $q=pV+U$, solutio problematis primo postulat, vt multiplicator M investigetur formulam $dx+V\,dy$ integrabilem reddens, quo inuento habebitur functio S earundem variabilium x et y, vt sit

$$S=\int M(dx+V\,dy).$$

Coroll. 2.

Coroll. 2.

148. In hunc finem confiderari conueniet aequationem differentialem $dx + V dy = 0$, haec enim fi integrari poterit, fimul inde colligi poteft multiplicator M, vt formula $M(dx + V dy)$ fiat verum differentiale cuiusdam functionis S quae propterea hinc inuenietur.

Coroll. 3.

149. Inuenta porro hac functione S, quantitas x per y et S exprimi debet, ita vt x aequetur functioni ipfarum y et S, quo valore in quantitate U fubftituto, quaeratur integrale $\int U dy = T$ fpectata S vt conftante, ficque obtinebitur T functio ipfarum y et S.

Coroll. 4.

150. Denique inuenta hac functione T fit $W = (\frac{dT}{dS})$ vnde tandem colligitur folutio problematis his duabus formulis contenta:

$$\frac{p}{M} = W + f' : S \text{ et } z = T + f : S$$

vbi cum S fit functio ipfarum x et y, pro z ftatim reperitur functio ipfarum x et y.

Coroll. 5.

151. Si U fit functio ipfius y tantum, non opus eft illa expreffione ipfius x per y et S, fed $T = \int U dy$ erit quoque functio ipfius y tantum, hinc

hinc $W = \left(\frac{dT}{ds}\right) = 0$. Hic autem casus manifesto reducitur ad praecedentem ponendo z loco $z - \int U dy$.

Exemplum 1.

152. *Si posito* $dz = pdx + qdy$ *debeat esse* $q = \frac{zz}{y} + \frac{y}{x}$, *indolem functionis* z *inuestigare.*

Hic ergo est

$$V = \frac{x}{y} \text{ et } U = \frac{y}{x};$$

vnde ob

$$dx + V dy = dx + \frac{x\,dy}{y}$$

erit multiplicator $M = y$, et $dS = y dx + x dy$ hinc $S = xy$, sicque habebitur

$$x = \frac{S}{y} \text{ et } U = \frac{yy}{S}.$$

Iam erit

$$T = \int U dy = \int \frac{yy\,dy}{S} = \frac{y^3}{3S} \text{ et } W = \frac{-y^3}{3SS}.$$

Quare pro solutione huius exempli habebimus

$$\frac{p}{y} = \frac{-y^3}{3SS} + f' : S \text{ et } z = \frac{y^3}{3S} + f : S$$

seu ob $S = xy$ erit

$$z = \frac{yy}{3x} + f : xy.$$

Exemplum 2.

153. *Si posito* $dz = pdx + qdy$ *debeat esse* $px + qy = n\sqrt{(xx + yy)}$ *indolem functionis* z *investigare.*

Cum hic fit $q = \frac{-px}{y} + \frac{n}{y}\sqrt{(xx+yy)}$, erit

$$V = \frac{-x}{y} \text{ et } U = \frac{n}{y}\sqrt{(xx+yy)}.$$

Ergo $dS = M(dx - \frac{xdy}{y})$, quare capiatur $M = \frac{1}{y}$ vt fiat

$$dS = \frac{dx}{y} - \frac{xdy}{yy} \text{ et } S = \frac{x}{y}.$$

Hinc oritur

$$x = Sy \text{ et } U = n\sqrt{(1+SS)};$$

ideoque posito S constante erit

$$T = \int U dy = ny\sqrt{(1+SS)} \text{ et } W = \left(\frac{dT}{dS}\right) = \frac{nyS}{\sqrt{(1+SS)}};$$

ita vt solutio nostrae quaestionis fit

$$py = \frac{nyS}{\sqrt{(1+SS)}} + f':S \text{ et } z = ny\sqrt{(1+SS)} + f:S.$$

Cum igitur fit $S = \frac{x}{y}$, erit

$$z = n\sqrt{(xx+yy)} + f:\frac{x}{y};$$

vbi $f:\frac{x}{y}$ denotat functionem quamcunque nullius dimensionis ipsarum x et y.

Exemplum 3.

154. *Si posito* $dz = pdx + qdy$ *debeat esse* $pxx + qyy = nxy$ *functionis* z *indolem inuestigare.*

Cum fit $q = \frac{-pxx}{yy} + \frac{nx}{y}$ erit

$$V = \frac{-xx}{yy} \text{ et } U = \frac{nx}{y}.$$

Quare

Quare ob $dS = M(dx - \frac{xx\,dy}{yy})$, capiatur $M = \frac{1}{xx}$, vt fiat $S = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$. Hinc erit

$$\frac{1}{x} = \frac{1}{y} - S \quad \text{et} \quad x = \frac{y}{1 - Sy},$$

ideoque $U = \frac{n}{1 - Sy}$. Sumto igitur S constante habebimus

$$T = \int \frac{n\,dy}{1 - Sy} = -\frac{n}{S} l(1 - Sy) \text{ et}$$
$$W = +\frac{n}{SS} l(1 - Sy) + \frac{ny}{S(1 - Sy)}.$$

Consequenter ob

$$S = \frac{x-y}{xy} \quad \text{et} \quad 1 - Sy = \frac{y}{x},$$

solutio praebet

$$z = \frac{-nxy}{x-y} l\frac{y}{x} + f : \frac{x-y}{xy}.$$

Scholion.

155. Ex solutione huius problematis etiam haec quaestio latius patens resolui potest. Sint P, Q item V, U functiones quaecunque datae ipsarum x et y, et quaeri oporteat functionem z vt sit

$$dz = P dx + Q dy + L(V dx + U dy)$$

seu quod eodem redit, functio L inuestigari debet, vt ista formula differentialis integrationem admittat. Ad hoc praestandum quaeratur primo multiplicator M formulam $V dx + U dy$ integrabilem efficiens, ponaturque $dS = M(V dx + U dy)$, vnde functio S

reperietur per x et y expressa. Ex ea quaeratur valor ipsius x per y et S expressus; et cum sit

$$dz = P\,dx + Q\,dy + \frac{L\,dS}{M},$$

hic vbique loco x valor ille substituatur; fit autem inde $dx = E\,dy + F\,dS$, vnde etiam E et F innotescent, eritque

$$dz = EP\,dy + Q\,dy + FP\,dS + \frac{L\,dS}{M}$$

sumatur quantitas S pro constante sitque

$$T = \int (EP + Q)\,dy \text{ erit}$$

$$z = T + f:S,$$

quod quidem ad solutionem sufficit; sed ad L inveniendum, differentietur haec expressio:

$$dz = (EP + Q)\,dy + dS.\left(\frac{dT}{dS}\right) + dS f':S$$

ac necesse est fiat

$$FP + \frac{L}{M} = \left(\frac{dT}{dS}\right) + f':S$$

ideoque

$$L = -FMP + M\left(\frac{dT}{dS}\right) + Mf':S.$$

Ceterum ob permutabilitatem ipsarum p, x et q, y etiam hinc sequentia problemata resolui possunt, quae propterea strictim percurram.

Proble-

Problema 24.

156. Si posito $dz = pdx + qdy$ requiratur vt sit $q = Vx + U$, existente tam V quam U functione quacunque data ipsarum p et y, inuestigare indolem functionis quaesitae z.

Solutio.

Vtamur formula

$$z = px + \int (qdy - xdp),$$

et cum loco q valore substituto sit

$$\int (qdy - xdp) = \int Vxdy - xdp + Udy)$$

quam formulam integrabilem reddi oportet. Sit ea breuitatis gratia $\mathfrak{Z}$, et cum sit

$$d\mathfrak{Z} = x(Vdy - dp) + Udy$$

quaeratur primo multiplicator M formulam $Vdy - dp$ integrabilem reddens, ponaturque

$$M(Vdy - dp) = dS,$$

sicque S dabitur per y et p; vnde p eliciatur per y et S expressum, quo valore ibi substituto erit

$$d\mathfrak{Z} = \frac{x\,dS}{M} + U\,dy.$$

Iam sumto S constante sumatur integrale

$\int U\,dy = T + f:S$, eritque

$\frac{x}{M} = \left(\frac{dT}{dS}\right) + f':S$ et $\mathfrak{Z} = T + f:S.$

Solutio igitur per binas variabiles y et S ita se habebit

$$x = \mathrm{M}\left(\frac{d\mathrm{T}}{d\mathrm{S}}\right) + \mathrm{M}f^{\prime}:\mathrm{S} \quad \text{et} \quad z = px + \mathrm{T} + f:\mathrm{S}$$

vbi nunc quidem S per p et y datur.

Problema 25.

157. Si posito $dz = pdx + qdy$ requiratur vt sit $p = \mathrm{V}y + \mathrm{U}$ existentibus V et U functionibus datis ipsarum x et q, indolem functionis z inuestigare.

Solutio.

Vtamur iam forma

$$z = qy + \int (p\,dx - y\,dq),$$

ponaturque formula ad integrationem perducenda

$$\int (p\,dx - y\,dq) = \mathfrak{v}.$$

Hinc pro p valorem assumtum substituendo erit

$$d\mathfrak{v} = \mathrm{V}y\,dx + \mathrm{U}\,dx - y\,dq = y(\mathrm{V}\,dx - dq) + \mathrm{U}\,dx.$$

Quaeramus multiplicatorem M vt fiat

$$\mathrm{M}(\mathrm{V}\,dx - dq) = d\mathrm{S}$$

ac tam M quam S erunt functiones ipsarum x et q, ex quarum posteriori valor ipsius q per x et S expressus eliciatur, in sequenti operatione pro q substituendus. Scilicet cum nunc sit

$$d\mathfrak{v} = \frac{y\,d\mathrm{S}}{\mathrm{M}} + \mathrm{U}\,dx,$$

sumto

sumto S constante quaeratur $T = \int U dx$, sitque

$$\mathfrak{z} = T + f:S,$$

vnde colligitur

$$\frac{z}{M} = \left(\frac{dT}{dS}\right) + f':S \text{ et } z = qy + T + f:S.$$

ac nunc quidem pro S valorem in x et q restituere licet.

Problema 26.

158. Si posito $dz = pdx + qdy$ requiratur vt sit $y = Vx + U$ existentibus V et U functionibus quibuscunque datis ipsarum p et q, indolem functionis z in genere inuestigare.

Solutio.

Hic vtendum est formula

$$z = px + qy - \int (xdp + ydq)$$

statuatur $\int (xdp + ydq) = \mathfrak{z}$, eritque pro y valorem praescriptum substituendo

$$d\mathfrak{z} = xdp + Vxdq + Udq.$$

Quaeratur iam multiplicator M formulam $dp + Vdq$ integrabilem reddens, sitque

$$M(dp + Vdq) = dS,$$

vbi M et S per p et q dabuntur; et ex posteriori eliciatur valor ipsius p per q et S expressus, quo deinceps vti oportet. Scilicet cum sit

$$d\mathfrak{z} = \frac{xdS}{M} + Udq,$$

sumto

ſumto S conſtante integretur formula Udq, ſitque $T = \int U dq$, erit $\mathfrak{z} = T + f:S$ hincque

$$\frac{x}{M} = \left(\frac{dT}{dS}\right) + f':S \text{ et } z = px + qy - T - f:S.$$

Omnia ergo per p et q, vnde M, S et T cum $\left(\frac{dT}{dS}\right)$ dantur, ita determinabuntur vt ſit

$$x = M\left(\frac{dT}{dS}\right) + Mf':S; \; y = Vx + U \text{ et}$$

$$z = px + qy - T - f:S.$$

Exemplum.

159. *Si poſito* $dz = pdx + qdy$ *debeat eſſe* $px + qy = apq$ *indolem functionis* z *inueſtigare.*

Cum ergo ſit

$$y = \frac{-px}{q} + ap, \text{ erit}$$

$$V = \frac{-p}{q} \text{ et } U = ap.$$

Quia nunc eſſe debet

$$M\left(dp - \frac{pdq}{q}\right) = dS,$$

capiatur $M = \frac{1}{q}$, ſitque

$$S = \frac{p}{q} \text{ et } p = Sq:$$

Hinc $U = aSq$ et ſumto S conſtante

$$T = \int U dq = \tfrac{1}{2} aSqq,$$

ideoque $\left(\frac{dT}{dS}\right) = \tfrac{1}{2} aqq.$

Quo-

Quocirca pro solutione habebimus:

$$x = \tfrac{1}{2}aq + \tfrac{1}{q}f' : \tfrac{p}{q} \; ; \; y = \tfrac{1}{2}ap - \tfrac{p}{qq}f' : \tfrac{p}{q} \text{ et}$$

$$z = px + qy - \tfrac{1}{2}apq - f : \tfrac{p}{q} = \tfrac{1}{2}apq - f : \tfrac{p}{q}.$$

Per reductionem autem supra traditam habebimus:

$$y = (aq - x)\,F' : (qx - \tfrac{1}{2}aqq) \text{ et}$$

$$z = qy + F : (qx - \tfrac{1}{2}aqq).$$

Scholion.

160. Quatuor problemata haec coniunctim considerata admodum late patent, atque pro formula $dz = pdx + qdy$ omnes relationes inter p, q, x et y complectuntur, in quibus vel x et y, vel p et y, vel x et q, vel p et q nusquam vnam dimensionem superant. Ex quo saepe fieri potest, vt eadem quaestio per duo pluraue horum quatuor problematum resolui possit; veluti euenit in exemplo hoc postremo, in quo cum non solum x et y, sed etiam x et q, itemque p et y nusquam plus vna dimensione occupant, id ad tria praecedentia problemata referri queat, haecque conditio primo tantum problemati aduersatur. Quod si autem inter p, q, x, y haec relatio praescribatur, vt esse debeat

$$\alpha px + \beta qy + ap + bq + mx + ny + c = 0,$$

resolutio per omnia quatuor problemata aeque institui potest. Verum etiam resolutiones inde ortae, etiamsi forma discrepent, tamen per reductionem

 ante

ante expositam ad consensum reuocari possunt. At sequens casus latissime patens resolutionem quoque admittit, quem propterea euolui conueniet.

Problema 27.

161. Si posito $dz = pdx + qdy$ inter p, q et x, y eiusmodi relatio detur, vt functio quaedam ipsarum p et x aequetur functioni cuipiam ipsarum q et y; functionis z indolem in genere inuestigare.

Solutio.

Sit P functio illa ipsarum p et x, et Q functio illa ipsarum q et y, quae inter se aequales esse debent. Cum igitur sit $P = Q$, ponatur vtraque $= v$, vt sit $P = v$ et $Q = v$. Ex priori ergo p definire licebit per x et v, ex posteriori vero q per y et v; quo facto in formula $dz = pdx + qdy$, cum p sit functio ipsarum x et v, integretur pars pdx sumto v constante sitque $\int pdx = R$, simili modo cum q sit functio ipsarum y et v, integretur quoque altera pars qdy sumto v constante, sitque $\int qdy = S$; erit ergo $R =$ functioni ipsarum x et v, et $S =$ functioni ipsarum y et v. At sumto etiam v variabili sit

$$dR = pdx + Vdv \text{ et } dS = qdy + Udv,$$

vnde colligitur

$$dz = dR + dS - dv(V + U),$$

quae

quae forma quia integrabilis esse debet, oportet sit $V+U=f':v$. Quare solutio problematis his duabus aequationibus continebitur:

$$V+U=f':v \text{ et } z=R+S-f:v.$$

Scilicet cum p, R et V dentur per x et v; atque q, S et U per y et v, per aequationem priorem definitur v ex x et y qui valor in altera substitutus determinabit functionem quaesitam z per x et y.

Coroll. 1.

162. Quoties ergo q eiusmodi functioni ipsarum p, x, y aequari debet, vt inde aequatio formari possit, ex cuius altera parte tantum binae litterae x et p, ex altera tantum binae reliquae y et q reperiantur problema resolui poterit.

Coroll. 2.

163. Si functio illa binarum litterarum p et x, quam posui P, ita sit comparata, vt posita ea $=v$ inde facilius x per p et v definiri possit; tum vti conueniet formula

$$z=px+\int(q\,dy-x\,dp),$$

et euolutio perinde se habebit atque ante.

Coroll. 3.

164. Simili modo si ex functione altera $Q = v$, quantitas y facilius per q et v definiatur, resolutio ex forma

$$z = qy + \int (p\,dx - y\,dq)$$

erit petenda. Sin autem vtrumque eueniat, vt tam x per p et v quam y per q et v definiatur, vtendum erit formula:

$$z = px + qy - \int (x\,dp + y\,dq).$$

Scholion.

165. Problema hoc innumerabiles complectitur casus in praecedentibus non comprehensos; atque etiam eius solutio diuerso nititur fundamento. Interim tamen longissime adhuc distamus a solutione problematis generalis, cui hoc caput est destinatum et quo in genere solutio desideratur, si inter quaternas quantitates p, q, x, y aequatio quaecunque proponatur; quae autem ob defectum Analyseos ne sperari quidem posse videtur. Contentos ergo nos esse oportet, si quam plurimos casus resoluere docuerimus. Quo autem vis huius problematis magis perspiciatur aliquot exempla adiungamus.

Exemplum 1.

166. *Si posito* $dz = p\,dx + q\,dy$ *esse debeat* $q = \frac{xxyy}{a^4 p}$ *indolem functionis* z *inuestigare.*

Quia

Quia hic p, x et q, y separare licet, cum sit $\frac{aaq}{yy}=\frac{xx}{aap}$ ponatur $\frac{xx}{aap}=v=\frac{aaq}{yy}$, vnde p per x et v, et q per y et v ita definitur vt sit

$$p=\frac{xx}{aav} \quad \text{et} \quad q=\frac{vyy}{aa},$$

ideoque

$$dz=\frac{xxdx}{aav}+\frac{vyydy}{aa}.$$

Hinc colligimus

$$z=\frac{x^3}{3aav}+\frac{vy^3}{3aa}+\frac{1}{3aa}\int\left(\frac{x^3dv}{vv}-y^3dv\right)$$

sicque $\frac{x^3}{vv}-y^3$ debet esse functio ipsius v. Ac posito

$$\frac{x^3}{vv}-y^3=f':v \quad \text{seu} \quad y^3=\frac{x^3}{vv}-f':v \quad \text{erit}$$

$$z=\frac{1}{3aa}\left(\frac{x^3}{v}+vy^3+f:v\right).$$

Corollarium.

167. Hinc facillime v eliminatur, si ponatur $f':v=\frac{b^3}{vv}-c^3$ hincque $f:v=\frac{-b^3}{v}-c^3v$. Iam prior aequatio dat $y^3-c^3=\frac{x^3-b^3}{vv}$, vnde $vv=\frac{x^3-b^3}{y^3-c^3}$, et ob

$$3aaz=\frac{x^3+vvy^3-b^3-c^3vv}{v}=2v(y^3-c^3), \text{ erit}$$

$$z=\frac{2}{3aa}\sqrt{(x^3-b^3)(y^3-c^3)}.$$

Exemplum 2.

168. *Si posito* $dz=pdx+qdy$ *debeat esse* $q=\frac{1}{b}\sqrt{(xx+yy-aapp)}$ *inuestigare indolem functionis* z.

 Condi-

Conditio praescripta redit ad

$$bbqq-yy=xx-aapp=v$$

vnde elicimus

$$q=\tfrac{1}{b}\sqrt{(yy+v)} \text{ et } p=\tfrac{1}{a}\sqrt{(xx-v)}.$$

Nunc vero est

$$\int p\,dx=\tfrac{1}{a}\int dx\sqrt{(xx-v)}=\tfrac{1}{2a}x\sqrt{(xx-v)}-\tfrac{v}{2a}\int\tfrac{dx}{\sqrt{(xx-v)}}$$
$$=\tfrac{x}{2a}\sqrt{(xx-v)}-\tfrac{v}{2a}l(x+\sqrt{(xx-v)})=R;$$

simili modo est

$$\int q\,dy=\tfrac{y}{2b}\sqrt{(yy+v)}+\tfrac{v}{2b}l(y+\sqrt{(yy+v)})=S.$$

Quare cum sit

$$V=\left(\tfrac{dR}{dv}\right)=\tfrac{-x}{4a\sqrt{(xx-v)}}-\tfrac{1}{2a}l(x+\sqrt{(xx-v)})$$
$$+\tfrac{v}{4a(x+\sqrt{(xx-v)})\sqrt{(xx-v)}}$$

quae reducitur ad

$$V=-\tfrac{1}{4a}-\tfrac{1}{2a}l(x+\sqrt{(xx-v)})$$

similique modo

$$U=\left(\tfrac{dS}{dv}\right)=+\tfrac{1}{4b}+\tfrac{1}{2b}l(y+\sqrt{(yy+v)}).$$

vbi cum $V+U=f':v$ erit

$$\frac{a-b}{4ab}+L.\frac{(y+\sqrt{(yy+v)})^{\frac{1}{2b}}}{(x+\sqrt{(xx-v)})^{\frac{1}{2a}}}=f':v$$

vnde

vnde valor ipsius v per x et y determinatur. Ex quo tandem colligitur:

$$z = \frac{x}{2a} V(xx-v) + \frac{y}{2b} V(yy+v) + vL \frac{(y+V(yy+v))^{\frac{1}{2b}}}{(x+V(xx-v))^{\frac{1}{2a}}} - f:v$$

seu

$$z = \frac{x}{2a} V(xx-v) + \frac{y}{2b} V(yy+v) - \frac{(a-b)v}{4ab} + vf':v - f:v.$$

Scholion.

169. Haec solutio a formulis logarithmicis liberari potest hoc modo. Ponatur

$$f':v = lt + \frac{a-b}{4ab},$$

vt sit

$$t^{2ab} = \frac{(y + V(yy+v))^{a}}{(x + V(xx-v))^{b}}$$

vnde v datur per t. Tum vero fit $v = tF':t$, et ob $dvf'':v = \frac{dt}{t}$ erit

$$\int v\,dvf'':v = vf':v - f:v = \int \frac{v\,dt}{t} = F:t,$$

sicque erit

$$z = \frac{x}{2a} V(xx-v) + \frac{y}{2b} V(yy+v) - \frac{(a-b)v}{4ab} + F:t,$$

vbi est

$$v = tF':t \quad \text{et} \quad t^{2ab} = \frac{(y+V(yy+v))^{a}}{(x+V(xx-v))^{b}},$$

vnde

vnde t et v per x et y definiri potest. Hinc statim patet si capiatur $F':t=0$, fore $v=0$, $F:t=0$ et $z=\frac{xx}{2a}+\frac{yy}{2b}$; hincque $p=\frac{x}{a}$ et $q=\frac{y}{b}$, quo pacto vtique conditioni praescriptae satisfit. Ceterum haec ratio quantitates logarithmicas elidendi maxime est notatu digna et in aliis casibus vsum amplissimum habere potest.

Exemplum 3.

170. *Si posito* $dz=pdx+qdy$ *debeat esse* $x^m y^n = A p^\mu q^\nu$ *indolem functionis* z *inuestigare.*

Statuatur ergo

$$\frac{x^m}{p^\mu}=\frac{Aq^\nu}{y^n}=v^{\mu\nu},$$

et hinc deducitur

$$p=\frac{x^{\frac{m}{\mu}}}{v^\nu} \quad \text{et} \quad q=\frac{1}{a}y^{\frac{n}{\nu}}v^\mu$$

posito $A=a^\nu$. Vnde habebimus

$$\int p\,dx=\frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)v^\nu}+\frac{\mu\nu}{m+\mu}\int\frac{x^{\frac{m+\mu}{\mu}}\,dv}{v^{\nu+1}} \quad \text{et}$$

$$\int q\,dy=\frac{\nu y^{\frac{n+\nu}{\nu}}v^\mu}{(n+\nu)a}-\frac{\mu\nu}{(n+\nu)a}\int y^{\frac{n+\nu}{\nu}}v^{\mu-1}\,dv.$$

Quocirca

Quocirca erit

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)v^{\nu}} + \frac{\nu y^{\frac{n+\nu}{\nu}} v^{\mu}}{(n+\nu)a} + \frac{\mu\nu}{(m+\mu)(n+\nu)a}$$

$$\int dv \left(\frac{(n+\nu)ax^{\frac{m+\mu}{\mu}}}{v^{\nu+1}} - (m+\mu)y^{\frac{n+\nu}{\nu}} v^{\mu-1}\right)$$

ita vt si statuamus

$$\frac{x^{\frac{m+\mu}{\mu}}}{(m+\mu)v^{\nu+1}} - \frac{y^{\frac{n+\nu}{\nu}} v^{\mu-1}}{(n+\nu)a} = f':v$$

futurum sit

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)v^{\nu}} + \frac{\nu y^{\frac{n+\nu}{\nu}} v^{\mu}}{(n+\nu)a} + \mu\nu f:v.$$

Pro casu simplicissimo ponamus $f':v = 0$ et $f:v = 0$ eritque

$$y^{\frac{n+\nu}{\nu}} v^{\mu+\nu} = \frac{(n+\nu)a}{m+\mu} x^{\frac{m+\mu}{\mu}} \text{ et } v = \left(\frac{(n+\nu)ax^{\frac{m+\mu}{\mu}}}{(m+\mu)y^{\frac{n+\nu}{\nu}}}\right)^{\frac{1}{\mu+\nu}}$$

tum vero

$$z = \frac{1}{v^{\nu}}\left(\frac{\mu}{m+\mu}x^{\frac{m+\mu}{\mu}} + \frac{\nu}{(n+\nu)a} y^{\frac{n+\nu}{\nu}} v^{\mu+\nu}\right) \text{ seu}$$

$$z = \frac{(\mu+\nu)}{(m+\mu)v^{\nu}} x^{\frac{m+\mu}{\mu}} = (\mu+\nu)\left(\frac{x^{m+\mu}y^{n+\nu}}{(m+\mu)^{\mu}(n+\nu)^{\nu}A}\right)^{\frac{1}{\mu+\nu}}.$$

Problema 28.

171. Si posito $dz = pdx + qdy$, inter p, q et x, y eiusmodi detur relatio, vt p et q aequentur functionibus quibusdam ipsarum x, y et nouae variabilis v, explorare casus, quibus, indolem functionis z inuestigare licet.

Solutio.

Cum sit p functio ipsarum x, y et v, spectatis y et v vt constantibus, quaeratur integrale $\int pdx = P$, sitque sumtis omnibus variabilibus

$$dP = pdx + Rdy + Mdv,$$

vnde si pro pdx valor substituatur, erit

$$dz = dP + (q - R)dy - Mdv.$$

Quodsi iam eueniat vt $q - R$ sit tantum functio ipsarum y et v exclusa x, sumto v constante quaeratur $\int (q - R)dy = T$ sitque deinceps

$$dT = (q - R)dy + Vdv.$$

Hinc valor ipsius $(q - R)dy$ ibi substitutus dabit

$$dz = dP + dT - (M + V)dv.$$

quae forma quia integrabilis esse debet statuatur

$$M + V = f' : v \text{ eritque } z = P + T - f : v.$$

Ex operationibus autem susceptis dantur P, R, M per x, y et v, at T et V per y et v tantum; ac ac resolutio succedit, si modo in forma $q-R$ non amplius x continetur. Pari ratione solutio succedet, si M tantum per y et v detur; tum enim ex y constante quaeratur $\int M\,dv = L$, sitque

$$dL = M\,dv + N\,dy \text{ erit}$$

$$dz = dP + (q-R+N)\,dy - dL$$

ponique conueniet

$$q - R + N = f':y$$

vt fiat

$$z = P - L + f:y.$$

Simili modo ab altera parte $\int q\,dy$ calculum incipere et prosequi licet.

Introducendo autem functionem ipsarum x, y et v indefinitam K negotium generalius confici poterit: Sit enim

$$dK = F\,dx + G\,dy + H\,dv,$$

ac consideretur haec forma:

$$dz + dK = (p+F)\,dx + (q+G)\,dy + H\,dv.$$

Nunc sumtis y et v constantibus quaeratur

$$\int(p+F)\,dx = P$$

sitque

$$dP = (p+F)\,dx + R\,dy + M\,dv,$$

vnde habetur:

$$dz + dK = dP + (q+G-R)\,dy + (H-M)\,dv.$$

 Quod

Quod si iam eueniat vt vel $q+G-R$ vel $H-M$ tantum binas variabiles y et v exclusa x contineat, resolutio vt ante est ostensum, absolui poterit.

Problema 29.

172. Si posito $dz=pdx+qdy$ relatio detur inter binas formulas differentiales p, q et binas variabiles x et z, vel y et z, solutionem problematis quatenus fieri potest, perficere.

Solutio.

Ponamus relationem dari inter p, q et x, z; atque hunc casum facile ad praecedentem reuocare licet. Consideretur enim haec formula

$$dy=\frac{dz-pdx}{q}$$

ex principali deriuata; voceturque

$$\frac{1}{q}=m \text{ et } \frac{p}{q}=-n,$$

vt habeatur

$$dy=mdz+ndx; \text{ et ob}$$

$$q=\frac{1}{m} \text{ et } p=-\frac{n}{m},$$

relatio proposita versabitur inter quaternas quantitates m, n, z et x ideoque quaestio omnino similis est earum, quas antea tractauimus, hoc tantum discrimine, quod hic quantitas y definiatur, cum ante esset z inuestigata. Quoniam autem ista determinatio per aequationes absoluitur, perinde est vtrum tandem inde z, an y elicere velimus. Quodsi ergo

hac

hac reductione facta quaestio in casus ante pertractatos incidat, methodis quoque expositis resolui poterit.

Exemplum.

173. *Si posito* $dz = pdx + qdy$ *debeat esse* $qxz = aap$, *indolem functionis* z *inuestigare.*

Consideretur formula $dy = \frac{dz}{q} - \frac{pdx}{q}$. Iam quia $\frac{p}{q} = \frac{xz}{aa}$ erit

$$dy = \frac{dz}{q} - \frac{xzdx}{aa} \text{ et } y = \int\left(\frac{dz}{q} - \frac{xzdx}{aa}\right) \text{ at est}$$

$$\int\frac{xzdx}{aa} = \frac{xxz}{2aa} - \int\frac{xxdz}{2aa}; \text{ ergo}$$

$$y = \int dz\left(\frac{1}{q} + \frac{xx}{2aa}\right) - \frac{xxz}{2aa}.$$

Ponatur ergo

$$\frac{1}{q} + \frac{xx}{2aa} = f':z \text{ erit } y = \frac{-xxz}{2aa} + f:z$$

ex qua aequatione vtique z per x et y definitur. Si pro casu simpliciori sumamus $f:z = b + \alpha z$ erit

$$y - b = \left(\alpha - \frac{xx}{2aa}\right)z \text{ et } z = \frac{2aa(y-b)}{2\alpha aa - xx};$$

et sumtis $\alpha = 0$ et $b = 0$ pro casu simplicissimo erit $z = \frac{-2aay}{xx}$. Hinc autem fit

$$p = +\frac{4aay}{x^3} \text{ et } q = \frac{-2aa}{xx}. \text{ Ergo}$$

$$\frac{p}{q} = -\frac{2y}{x} \text{ et } \frac{xz}{aa} = \frac{-2y}{x}.$$

CAPVT VI.

DE RESOLVTIONE AEQVATIONVM QVIBVS RELATIO INTER BINAS FORMVLAS DIFFERENTIALES $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$, ET OMNES TRES VARIABILES x, y, z QVAECVNQVE DATVR.

Problema 30.

174.

Si posito $dz = pdx + qdy$, debeat esse $nz = px + qy$ indolem functionis z in genere inuestigare.

Solutio.

Ope relationis datae elidatur vel p vel q: Scilicet cum sit $q = \frac{nz}{y} - \frac{px}{y}$ erit

$$dz = pdx + \frac{nzdy}{y} - \frac{pxdy}{y},$$

quae aequatio in hanc formam transfundatur

$$dz - \frac{nzdy}{y} = p\left(dx - \frac{xdy}{y}\right) = pyd.\frac{x}{y}.$$

Vt

Vt prius membrum $dz - \frac{nzdy}{y}$ integrabile reddatur, multiplicetur aequatio per $\frac{1}{z}$. funct. $\frac{z}{y^n}$, seu particulariter per $\frac{1}{y^n}$ eritque $d.\frac{z}{y^n} = py^{1-n}\, d.\frac{x}{y}$. Quo facto euidens est poni debere $py^{1-n} = f':\frac{x}{y}$ vt fiat $\frac{z}{y^n} = f:\frac{x}{y}$ seu $z = y^n f:\frac{x}{y}$. Vnde patet fore z functionem homogeneam ipsarum x et y dimensionum numero existente $= n$.

Si in genere aequatio multiplicetur per $\frac{1}{z}$. funct. $\frac{z}{y^n}$, erit partis prioris integrale $F:\frac{z}{y^n}$, pro parte autem altera si ponatur $\frac{py}{z}$ funct. $\frac{z}{y^n} = f':\frac{x}{y}$ erit $F:\frac{z}{y^n} = f:\frac{x}{y}$ atque vt ante $\frac{z}{y^n}$ aequabitur functioni cuicunque ipsius $\frac{x}{y}$.

Coroll. 1.

175. Cum z aequetur functioni homogeneae n dimensionum ipsarum x et y, erunt p et q functiones $n-1$ dimensionum. Scilicet cum sit $z = y^n f:\frac{x}{y}$ erit

$$p = y^{n-1} f':\frac{x}{y} \quad \text{et} \quad q = ny^{n-1} f:\frac{x}{y} - xy^{n-2} f':\frac{x}{y},$$

vnde fit manifesto $nz = px + qy$.

Coroll. 2.

Coroll. 2.

176. Si p et q fuerint functiones $n-1$ dimensionum ipsarum x et y, ac formula $pdx+qdy$ sit integrabilis seu $\left(\frac{dp}{dy}\right)=\left(\frac{dq}{dx}\right)$, tum integrale certo erit $\frac{px+qy}{n}$, quae proprietas nonnunquam insignem vsum habere potest.

Scholion.

177. Fundamentum huius solutionis in hoc consistit quod aequatio integranda in duas partes resoluatur, quarum vtraque ope certi multiplicatoris integrabilis reddi queat, vnde deinceps vna quantitas variabilis, cuius differentiale in aequatione non occurrit, determinetur. Hinc aequatio nostra

$$dz-\frac{nzdy}{y}=p\left(dx-\frac{xdy}{y}\right),$$

etiam ita repraesentari potest

$$\frac{dx}{y}-\frac{xdy}{yy}=\frac{1}{py}\left(dz-\frac{nzdy}{y}\right)=\frac{y^{n-1}}{p}\left(\frac{dz}{y^n}-\frac{nzdy}{y^{n+1}}\right) \text{ seu}$$

$$d.\frac{x}{y}=\frac{y^{n-1}}{p}\,d.\frac{z}{y^n}.$$

Sit ergo

$$\frac{y^{n-1}}{p}=F':\frac{z}{y^n} \text{ eritque}$$

$$\frac{x}{y}=F:\frac{z}{y^n} \text{ ac vicissim } \frac{z}{y^n}=f:\frac{x}{y} \text{ vt ante.}$$

Possumus

Possumus etiam statim z ex calculo elidere; cum enim sit

$$nz = px + qy \text{ erit}$$

$$ndz = pdx + qdy + xdp + ydq.$$

At est

$$ndz = npdx + nqdy$$

per hypothesin, ideoque

$$(n-1)pdx - xdp + (n-1)qdy - ydq = 0 \text{ seu}$$

$$x^n\left(\frac{(n-1)pdx}{x^n} - \frac{dp}{x^{n-1}}\right) + y^n\left(\frac{(n-1)qdy}{y^n} - \frac{dq}{y^{n-1}}\right) = 0$$

quae reducitur ad hanc formam:

$$-x^n d.\frac{p}{x^{n-1}} - y^n d.\frac{q}{y^{n-1}} = 0 \text{ seu}$$

$$d.\frac{q}{y^{n-1}} = -\frac{x^n}{y^n} d.\frac{p}{x^{n-1}}.$$

Statuatur

$$\frac{x^n}{y^n} = -f' : \frac{p}{x^{n-1}} \text{ erit } \frac{q}{y^{n-1}} = f : \frac{p}{x^{n-1}}.$$

Vel posito $\frac{x}{y} = v$, si ob $v^n = -f' : \frac{p}{x^{n-1}}$ reciproce ponatur

$$\frac{p}{x^{n-1}} = u = \frac{1}{v^{n-1}} F' : v,$$

vt sit

$$f':u = -v^n,$$

reperietur

$$\int du f':u = f:u = nF:v - vF':v.$$

Hinc

$$p = \frac{x^{n-1}}{y^{n-1}} F':v = y^{n-1} F':\frac{x}{y} \text{ et}$$

$$q = y^{n-1} f:u = n y^{n-1} F:\frac{x}{y} - x y^{n-2} F':\frac{x}{y};$$

ideoque

$$nz = px + qy = n y^n F:\frac{x}{y} \text{ seu } z = y^n F:\frac{x}{y}$$

vt ante.

Problema 31.

178. Si posito $dz = pdx + qdy$, debeat esse $\alpha px + \beta qy = nz$ indolem functionis z inuestigare.

Solutio.

Ex conditione praescripta eliciatur vt ante

$$q = \frac{nz}{\beta y} - \frac{\alpha px}{\beta y} \text{ eritque}$$

$$dz - \frac{nz\,dy}{\beta y} = pdx - \frac{\alpha px\,dy}{\beta y},$$

quae aequatio per $y^{\frac{n}{\beta}}$ diuisa dat

$$d\frac{z}{y^{n:\beta}} = \frac{p}{y^{n:\beta}}\left(dx - \frac{\alpha x\,dy}{\beta y}\right) = \frac{py^{\alpha:\beta}}{y^{n:\beta}}\, d\frac{x}{y^{\alpha:\beta}}.$$

Quodsi

Quod si ergo ponamus

$$p y^{(\alpha-n):\beta} = f' : \frac{x}{y^{\alpha:\beta}}$$

habebimus solutionem

$$z = y^{n:\beta} f : \frac{x}{y^{\alpha:\beta}}.$$

At functio ipsius $\frac{x}{y^{\alpha:\beta}}$ reducitur ad functionem ipsius $\frac{x^\beta}{y^\alpha}$, vnde z etiam ita per x et y determinatur vt sit

$$z = y^{n:\beta} f : \frac{x^\beta}{y^\alpha},$$

vel etiam

$$z^{\frac{1}{n}} = y^{\frac{1}{\beta}} f : \frac{x^{\frac{1}{\alpha}}}{y^{\frac{1}{\beta}}}.$$

Quodsi ergo quantitates $x^{\frac{1}{\alpha}}$ et $y^{\frac{1}{\beta}}$ vnam dimensionem constituere censeantur, $z^{\frac{1}{n}}$ aequabitur earundem functioni vnius dimensionis, ipsa autem quantitas z earundem functioni n dimensionum. Vel sumta pro z functione quacunque homogenea n dimensionum binarum variabilium t et u, scribatur deinde $t = x^{\frac{1}{\alpha}}$ et $u = y^{\frac{1}{\beta}}$ ac prodibit functio conueniens pro z.

Problema 32.

179. Si posito $dz = pdx + qdy$ debeat esse $Z = pX + qY$ denotante Z functionem ipsius z, X ipsius x et Y ipsius y indolem functionis Z in genere inuestigare.

Solutio.

Ex conditione praescripta elicitur $q = \frac{Z}{Y} - \frac{pX}{Y}$, qui valor substitutus praebet

$$dz - \frac{Zdy}{Y} = p\left(dx - \frac{Xdy}{Y}\right),$$

hincque

$$\frac{dz}{Z} - \frac{dy}{Y} = \frac{p}{Z}\left(dx - \frac{Xdy}{Y}\right) = \frac{pX}{Z}\left(\frac{dx}{X} - \frac{dy}{Y}\right),$$

vbi iam resolutio est manifesta. Statuatur scilicet

$$\frac{pX}{Z} = f' : \left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right)$$

eritque

$$\int\frac{dz}{Z} - \int\frac{dy}{Y} = f : \left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right),$$

vnde valor ipsius z per x et y definitur.

Coroll. 1.

180. Hic ergo z ita per x et y definiri debet, vt si X, Y et Z datae sint functiones sigillatim ipsarum x, y et z fiat:

$$X\left(\frac{dz}{dx}\right) + Y\left(\frac{dz}{dy}\right) = Z;$$

cuius

cuius ergo aequationis resolutionem hic inuenimus hac aequatione finita contentam:

$$\int\frac{dz}{Z}=\int\frac{dy}{Y}+f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right).$$

Coroll. 2.

181. Quemadmodum autem hic valor conditioni problematis satisfaciat, ex eius differentiatione statim patet. Cum enim sit:

$$\frac{dz}{Z}=\frac{dy}{Y}+\left(\frac{dx}{X}-\frac{dy}{Y}\right)f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right) \text{ erit}$$

$$\left(\frac{dz}{dx}\right)=\frac{Z}{X}f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right) \text{ et}$$

$$\left(\frac{dz}{dy}\right)=\frac{Z}{Y}-\frac{Z}{Y}f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)$$

vnde fit

$$X\left(\frac{dz}{dx}\right)+Y\left(\frac{dz}{dy}\right)=Z.$$

Scholion.

182. Solutio ergo eodem modo, vt fecimus sine introductione nouarum litterarum p et q absolui potest, retinendo earum loco valores differentiales $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$; facilius autem singulae litterae scribuntur, calculusque fit breuior. Ceterum ex hoc problematum genere, vbi omnes tres variabiles x, y et z praeter binos valores differentiales p et q in determinationem ingrediuntur, paucissima resoluere licet; ac praeter hoc, quod tractavimus vix vnum aut alterum insuper adiungere

 pote-

poterimus. Vnde hic insignia adhuc calculi incrementa desiderantur. Quo autem huius problematis vis penitius inspiciatur, nonnulla exempla subiungamus.

Exemplum 1.

183. *Si posito* $dz = pdx + qdy$ *debeat esse* $zz = pxx + qyy$, *indolem functionis* z *in genere investigare.*

Hic ergo est $Z = zz$, $X = xx$, et $Y = yy$; vnde habemus

$$\int \frac{dx}{X} = -\frac{1}{x};\ \int \frac{dy}{Y} = -\frac{1}{y} \text{ et } \int \frac{dz}{Z} = -\frac{1}{z}$$

quibus valoribus substitutis pro solutione adipiscimur:

$$-\frac{1}{z} = -\frac{1}{y} + f:\left(\frac{1}{y} - \frac{1}{x}\right), \text{ seu}$$

$$z = \frac{y}{1 - yf:\left(\frac{1}{y} - \frac{1}{x}\right)}$$

sumatur ergo functio quaecunque quantitatis

$$\frac{1}{y} - \frac{1}{x} = \frac{x - y}{xy},$$

quae si ponatur V erit $z = \frac{y}{1 - Vy}$.

Veluti si ponamus $V = \frac{n}{y} - \frac{n}{x}$, erit

$$\frac{1}{z} = \frac{1}{y} - \frac{n}{y} + \frac{n}{x} = \frac{ny - (n-1)x}{xy},$$

hincque $z = \frac{xy}{ny - (n-1)x}$, vnde fit

$$p = \left(\frac{dz}{dx}\right) = \frac{nyy}{(ny - (n-1)x)^2} \text{ et } q = \left(\frac{dz}{dy}\right) = \frac{-(n-1)xx}{(ny - (n-1)x)^2}$$

sicque $pxx + qyy = \frac{xxyy}{(ny - (n-1)x)^2} = zz$.

Exem-

Exemplum 2.

184. *Si posito* $dz = pdx + qdy$ *debeat esse* $\frac{n}{z} = \frac{p}{x} + \frac{q}{y}$, *indolem functionis* z *inuestigare.*

Cum hic sit

$$X = \tfrac{1}{x};\; Y = \tfrac{1}{y} \text{ et } Z = \tfrac{n}{z} \text{ erit}$$

$$\int \tfrac{dx}{X} = \tfrac{1}{2}xx;\; \int \tfrac{dy}{Y} = \tfrac{1}{2}yy \text{ et } \int \tfrac{dz}{Z} = \tfrac{1}{2n}zz$$

vnde solutio ita erit comparata:

$$\tfrac{1}{2n}zz = \tfrac{1}{2}yy + f:(xx - yy) \text{ siue}$$

$$zz = nyy + f:(xx - yy)$$

non enim est necesse functionem per $2n$ multiplicari, cum ea omnes operationes iam per se inuoluat.

Si pro hac functione sumatur $\alpha(xx - yy)$ habebitur solutio particularis

$zz = \alpha xx + (n - \alpha)yy$ et $z = \surd(\alpha xx + (n - \alpha)yy)$

hincque

$$p = \left(\tfrac{dz}{dx}\right) = \frac{\alpha x}{\surd(\alpha xx + (n - \alpha)yy)} \text{ et}$$

$$q = \left(\tfrac{dz}{dy}\right) = \frac{(n - \alpha)y}{\surd(\alpha xx + (n - \alpha)yy)}$$

seu $\frac{p}{x} = \frac{\alpha}{z}$ et $\frac{q}{y} = \frac{n - \alpha}{z}$, ideoque $\frac{p}{x} + \frac{q}{y} = \frac{n}{z}$.

Problema

Problema 32.

185. Si posito $dz = p\,dx + q\,dy$ debeat esse $q = pT + V$ existente T functione quacunque ipsarum x et y, ac V functione ipsarum y et z, investigare indolem functionis z.

Solutio.

Substituto loco q valore praescripto, huic aequationi inducatur forma:

$$dz - V\,dy = p(dx + T\,dy).$$

Cum iam V tantum binas variabiles y et z involuat, dabitur multiplicator M prius membrum $dz - V\,dy$ integrabile reddens; ponatur ergo

$$M(dz - V\,dy) = dS.$$

Simili modo quia T tantum x et y continet, dabitur multiplicator L membrum quoque posterius $dx + T\,dy$ integrabile efficiens; sit igitur

$$L(dx + T\,dy) = dR,$$

ita vt nunc sint R et S functiones cognitae, illa ipsarum x et y, haec vero ipsarum y et z. Hinc nostra aequatio induet hanc formam

$$\frac{dS}{M} = \frac{p\,dR}{L} \text{ seu } dS = \frac{pM\,dR}{L},$$

cuius integrabilitas necessario postulat vt sit $\frac{pM}{L}$ functio ipsius R. Ponamus ergo

$$\frac{pM}{L} = f' : R \text{ eritque } S = f : R$$

qua aequatione relatio inter z et x, y definitur.

Coroll. 1.

Coroll. 1.

186. In hoc problemate praecedens tanquam casus particularis continetur: cum enim ibi esset $Z = pX + qY$ erit $q = -\frac{X}{Y}p + \frac{Z}{Y}$, ideoque huius problematis applicatione facta fit $T = -\frac{X}{Y}$ et $V = \frac{Z}{Y}$.

Coroll. 2

187. Quanquam autem hoc problema infinite latius patet quam praecedens, arctissimis tamen adhuc limitibus continetur, neque eius ope vel hunc casum simplicissimum $z = py + qx$ resoluere licet.

Scholion.

188. Omnino est haec forma $z = py + qx$ digna notatu quod nulla ratione hactenus cognita resolui posse videtur. Siue enim inde eliciatur $q = \frac{z - py}{x}$, vnde fit

$$dz - \frac{z\,dy}{x} = p\left(dx - \frac{y\,dy}{x}\right)$$

siue simili modo p nulla via ad solutionem patet; cuius difficultatis causa in hoc manifesto est posita, quod formula $dz - \frac{z\,dy}{x}$ nullo multiplicatore integrabilis reddi potest; seu quod haec aequatio $dz - \frac{z\,dy}{x} = 0$ plane est impossibilis, cum x perinde sit variabilis atque y et z. Supra scilicet iam notaui non omnes aequationes differentiales inter ternas variabiles esse possibiles, simulque characterem possibilitatis exhi-

exhibui, qui pro tali forma

$$dz + P\,dx + Q\,dy = 0$$

huc reducitur vt sit

$$P\left(\frac{dQ}{dz}\right) - Q\left(\frac{dP}{dz}\right) = \left(\frac{dQ}{dx}\right) - \left(\frac{dP}{dy}\right)$$

nostro iam casu est $P = 0$ et $Q = \frac{-z}{x}$, vnde hic character dat $0 = \frac{z}{xx}$, quod cum sit falsum, etiam aequatio illa $dz - \frac{z\,dy}{x} = 0$ est impossibilis, quod quidem per se est manifestum. Verum tamen pro hoc casu $z = py + qx$ solutio particularis est obuia scilicet $z = n(x+y)$, vnde fit $p = q = n$. Deinceps autem methodum dabimus ex huiusmodi solutione particulari generalem eruendi.

Exemplum 1.

189. *Si posito* $dz = p\,dx + q\,dy$ *debeat esse* $py + qx = \frac{nxz}{y}$, *indolem functionis z inuestigare.*

Cum hinc sit $q = -\frac{py}{x} + \frac{nz}{y}$ erit

$$T = \frac{-y}{x} \quad \text{et} \quad V = \frac{nz}{y}$$

vnde fit

$$dS = M\left(dz - \frac{nz\,dy}{y}\right) \text{ et } dR = L\left(dx - \frac{y\,dy}{x}\right).$$

Sumatur ergo $M = \frac{1}{y^n}$ vt fiat $S = \frac{z}{y^n}$ et $L = 2x$

vt

vt fiat $R = xx - yy$; Quocirca hanc adipiſcimur ſolutionem:

$$\frac{z}{y^n} = f:(xx-yy) \text{ ſeu } z = y^n f:(xx-yy).$$

Exemplum 2.

190. *Si poſito* $dz = pdx + qdy$ *debeat eſſe* $pxx + qyy = nyz$ *definire indolem functionis* z.

Cum ergo ſit $q = -\frac{pxx}{yy} + \frac{nz}{y}$, erit

$$T = -\frac{xx}{yy} \text{ et } V = \frac{nz}{y}$$

ſicque hic caſus in noſtro problemate continetur. Vnde colligi oportet:

$$dR = L\left(dx - \frac{xxdy}{yy}\right) \text{ et } dS = M\left(dz - \frac{nzdy}{y}\right).$$

Quare ſumto $L = \frac{1}{xx}$ fit $R = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$ et ſumto $M = \frac{1}{y^n}$, fit $S = \frac{z}{y^n}$, ideoque ſolutio prodit iſta:

$$\frac{z}{y^n} = f:\frac{x-y}{xy} \text{ et } z = y^n f:\frac{x-y}{xy}.$$

Problema 33.

191. Si poſito $dz = pdx + qdy$ debeat eſſe $p = qT + V$ exiſtente T functione ipſarum x et y, at V functione ipſarum x et z, indolem functionis z inueſtigare.

Solutio.

Simili modo vt ante si loco p valor praescriptus substituatur obtinebitur:

$$dz - V\,dx = q(dy + T\,dx).$$

Iam ob indolem functionum V et T sequentes integrationes instituere licebit:

$$M(dz - V\,dx) = dS \text{ et } N(dy + T\,dx) = dR$$

vnde fit

$$\frac{dS}{M} = \frac{q\,dR}{N} \text{ seu } dS = \frac{Mq}{N}\,dR.$$

Atque hinc facillime colligitur haec solutio:

$$\frac{Mq}{N} = f' : R \text{ et } S = f : R.$$

Problema 34.

192. Si posito $dz = p\,dx + q\,dy$ debeat esse $z = Mp + Nq$ existentibus M et N functionibus quibusuis binarum variabilium x et y; ex quadam solutione particulari, qua constat esse $z = V$, indolem functionis z in genere determinare.

Solutio.

Valor iste particularis V, qui est functio ipsarum x et y differentietur, sitque

$$dV = P\,dx + Q\,dy,$$

qui

qui valor quia loco z substitutus satisfacit, vbi fit $p = P$ et $q = Q$, erit per hypothesin

$$V = MP + NQ.$$

Iam generatim ponatur $z = Vf:T$ sitque

$$dT = R\,dx + S\,dy,$$

et nunc quaeri oportet hanc functionem T. Ex differentiatione autem eruimus:

$$p = \left(\tfrac{dz}{dx}\right) = Pf:T + VRf':T \text{ et}$$

$$q = \left(\tfrac{dz}{dy}\right) = Qf:T + VSf':T.$$

Quare cum sit

$$z = Mp + Nq = Vf:T \text{ erit}$$

$$Vf:T = (MP + NQ)f:T + V(MR + NS)f':T$$

et ob $V = MP + NQ$ per hypothesin habebitur

$$MR + NS = 0, \text{ hinc}$$

$$dT = R\left(dx - \tfrac{M\,dy}{N}\right).$$

Iam nosse non oportet R, sed sufficit considerari formulam $N\,dx - M\,dy$ quae ope multiplicatoris cuiusdam integrabilis reddi potest. Solutio ergo facillime huc redit, vt ex conditione praescripta $z = Mp + Nq$ formetur aequatio realis:

$$dT = R(N\,dx - M\,dy),$$

inuento enim multiplicatore idoneo R, per integrationem reperitur quantitas T, qua inuenta erit $z = Vf:T$.

Aliter.

Facilius valor generalis hoc modo inuenitur; ob valorem ipsius z cognitum V statuatur $z = Vv$, sitque $dv = rdx + sdy$; erit

$$p = Pv + Vr \text{ et } q = Qv + Vs,$$

ideoque

$$z = Mp + Nq = (MP + NQ)v + V(Mr + Ns) = Vv.$$

At est $V = MP + NQ$; ergo

$$Mr + Ns = 0 \text{ seu } s = -\frac{Mr}{N}.$$

Vnde fit

$$dv = r\left(dx - \frac{Mdy}{N}\right) = \frac{r}{N}(Ndx - Mdy).$$

Statuatur ergo idoneum multiplicatorem inuestigando

$$R(Ndx - Mdy) = dT \text{ erit, } dv = \frac{r}{NR} dT$$

ex quo colligitur

$$\frac{r}{NR} = f' : T \quad \text{et} \quad v = f : T$$

ita vt in genere sit vt ante $z = Vv$.

Coroll. 1.

193. Proposita ergo conditione $z = Mp + Nq$ vt sit $dz = pdx + qdy$ statim consideretur aequatio differentialis $R(Ndx - Mdy) = dT$, vnde tam multiplicator R quam inde integrale T reperitur; haecque operatio non pendet a valore particulari cognito V.

Coroll. 2.

Coroll. 2.

194. Inuenta autem quantitate T, si vndecunque innotuerit solutio particulariter satisfaciens $z = V$, erit solutio generalis $z = Vf:T$. Probe autem notetur ex solutione particulari generalem elici non posse nisi conditio praescripta sit huiusmodi $z = Mp + Nq$.

Exemplum 1.

195. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = py + qx$ *ex valore particulari* $z = x + y$ *generalem definire.*

Cum hic sit $M = y$ et $N = x$, habebimus hanc aequationem

$R(xdx - ydy) = dT$, hincque

$T = f:(xx - yy)$

ergo solutio generalis erit

$z = (x + y) f:(xx - yy)$.

Exemplum 2.

196. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = p(x + y) + q(y - x)$ *ex valore particulari* $z = \surd(xx + yy)$ *generalem inuenire.*

Ob $M = x + y$ et $N = y - x$ formula $Ndx - Mdy$ deducit ad hanc aequationem:

$R(ydx - xdx - xdy - ydy) = dT$.

Suma-

Sumatur $R = \frac{1}{xx+yy}$, vt fit

$$dT = \frac{ydx - xdy}{xx+yy} - \frac{xdx - ydy}{xx+yy} \text{ erit}$$

$$T = \text{Ang. tang. } \frac{x}{y} - \tfrac{1}{2}l(xx+yy).$$

Atque ex valore hoc dupliciter transcendente erit

$$z = \sqrt{(xx+yy)}f : T,$$

simulque patet nullum alium dari valorem particularem, qui sit algebraicus, praeter datum $z = \sqrt{(xx+yy)}$.

Exemplum 3.

197. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = p(\alpha x + \beta y) + q(\gamma x + \delta y)$ *ex inuento valore particulari* $z = V$, *indolem functionis* z *in genere definire.*

Hic est $M = \alpha x + \beta y$ et $N = \gamma x + \delta y$, vnde deducimur ad hanc aequationem:

$$R((\gamma x + \delta y)dx - (\alpha x + \beta y)dy) = dT$$

vbi ob formam homogeneam debet esse

$$R = \frac{1}{\gamma xx + (\delta - \alpha)xy - \beta yy}.$$

vt fit

$$dT = \frac{(\gamma x + \delta y)dx - (\alpha x + \beta y)dy}{\gamma xx + (\delta - \alpha)xy - \beta yy},$$

ad quod integrale inueniendum ponatur $y = ux$, ac prodibit

$$dT = \frac{dx}{x} - \frac{(\alpha + \beta u)du}{\gamma + (\delta - \alpha)u - \beta uu}$$

sit $\int \frac{(\alpha + \beta u)du}{\gamma + (\delta - \alpha)u - \beta uu} = lU$, erit $T = lx - lU$

et

et cum functio ipsius T sit etiam functio ipsius $\frac{z}{U}$ erit in genere $z = Vf : \frac{z}{U}$. Patet autem, cum U sit functio ipsius $u = \frac{y}{x}$ fore U functionem homogeneam nullius dimensionis ipsarum x et y, ideoque $\frac{z}{U}$ functionem vnius dimensionis.

Scholion.

198. Hoc ergo exemplo difficultas restat, quomodo solutio particularis $z = V$ obtineri queat; nisi enim vna saltem huiusmodi solutio particularis constet, solutio generalis ne absolui quidem potest. Pro hoc autem casu solutionem particularem sequenti modo elicere licet, qui cum aliquid singulare habeat, nullum est dubium, quin eius ope hoc calculi genus haud parum adiumenti sit consecuturum.

Problema 35.

199. Si posito $dz = pdx + qdy$ debeat esse

$$z = p(\alpha x + \beta y) + q(\gamma x + \delta y)$$

valorem particularem inuestigare, qui loco z substitutus huic conditioni satisfaciat.

Solutio.

Negotium hoc succedet, si pro z eiusmodi valorem quaeramus, qui sit functio nullius dimensionis ipsarum x et y, seu posito $y = ux$, qui sit functio

functio ipsius u tantum. Ponamus ergo

$z = f : u = f : \frac{y}{x}$, eritque

$f' : u = \frac{dz}{du}$; at ob

$du = \frac{dy}{x} - \frac{y\,dx}{xx}$ erit

$dz = \left(\frac{dy}{x} - \frac{u\,dx}{x}\right) f' : u$; hinc

$p = -\frac{u}{x} f' : u = -\frac{u\,dz}{x\,du}$ et $q = \frac{1}{x} f' : u = \frac{dz}{x\,du}$.

Quibus valoribus pro p et q substitutis, conditio praescripta praebet:

$$z = x(\alpha + \beta u)p + x(\gamma + \delta u)q = \frac{-u\,dz(\alpha+\beta u) + dz(\gamma+\delta u)}{du}$$

vnde fit

$$\frac{dz}{z} = \frac{du}{\gamma + (\delta - \alpha)u - \beta uu}.$$

Ponamus

$$\int \frac{du}{\gamma + (\delta - \alpha)u - \beta uu} = l\,V$$

vt fiat $z = V$ eritque V valor particularis pro z satisfaciens.

Coroll. 1.

200. Inuento hoc valore V praecedentis exempli ope solutio generalis facile inuenitur. Erit scilicet $z = V f : \frac{x}{U}$ existente

$$\frac{dU}{U} = \frac{(\alpha + \beta u)\,dv}{\gamma + (\delta - \alpha)u - \beta uu};$$

vnde patet quantitatem U ex ipso valore particulari V inueniri posse.

Coroll. 2.

Coroll. 2.

201. Erit enim

$$lU = -l\sqrt{(\gamma+(\delta-\alpha)u-\beta uu)}+\int\frac{\frac{1}{2}(\delta+\alpha)du}{\gamma+(\delta-\alpha)u-\beta uu}$$

ideoque

$$lU = -l\sqrt{(\gamma+(\delta-\alpha)u-\beta uu)}+\tfrac{1}{2}(\alpha+\delta)lV$$

siue

$$U=\frac{V^{\frac{1}{2}(\alpha+\delta)}}{\sqrt{(\gamma+(\delta-\alpha)u-\beta uu)}};\ \text{hinc}$$

$$\frac{x}{U}=\frac{\sqrt{(\gamma xx+(\delta-\alpha)xy-\beta yy)}}{V^{\frac{1}{2}(\alpha+\delta)}}.$$

Coroll. 3.

202. Quocirca inuento valore particulari $z=V$ vt sit

$$\frac{dV}{V}=\frac{du}{\gamma+(\delta-\alpha)u-\beta uu}\ \text{existente}\ u=\frac{y}{x};$$

erit valor generaliter satisfaciens:

$$z=Vf:\frac{\gamma xx+(\delta-\alpha)xy-\beta yy}{V^{\alpha+\delta}}=Vf:\frac{x(\gamma x+\delta y)-y(\alpha x+\beta y)}{V^{\alpha+\delta}}.$$

Coroll. 4.

202. Hinc colligitur alius valor particularis; qui semper est algebraicus, erit is scilicet

$$z=(x(\gamma x+\delta y)-y(\alpha x+\beta y))^{\frac{1}{\alpha+\delta}}$$

Exemplum 1.

205. *Si posito* $dz = pdx + qdy$ *debeat esse* $nz = py - qx$; *indolem functionis* z *inuestigare.*

Comparatione cum forma nostra generali instituta fit

$$\alpha = 0;\ \beta = \tfrac{1}{n};\ \gamma = -\tfrac{1}{n};\ \delta = 0.$$

Hic ergo casus ob $\delta = -\alpha$ pertinet ad §. praecedentem vnde fit

$$lV = \int \frac{n\,du}{-1-uu} = -n \text{ Ang. tang. } u.$$

Cum igitur sit $u = \frac{y}{x}$, forma generalis est

$$z = e^{-n \text{Ang. tang.} \frac{y}{x}} f:(xx+yy).$$

Exemplum 2.

206. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = p(x+y) - q(x+y)$ *indolem functionis* z *inuestigare.*

Comparatione facta fit

$$\alpha = 1;\ \beta = 1;\ \gamma = -1;\ \delta = -1$$

hincque

$$lV = \int \frac{du}{-1-2u-uu} = \frac{1}{1+u} \text{ et } V = e^{\frac{1}{1+u}}$$

et solutio generalis est

$$z = e^{\frac{x}{x+y}} f:(x+y).$$

 Exem-

Exemplum 3.

207. *Si poſito* $dz = pdx + qdy$ *debeat eſſe* $z = p(x-2y) + q(2x-3y)$ *indolem functionis* z *inveſtigare.*

Cum ergo hic ſit

$$\alpha = 1;\ \beta = -2;\ \gamma = 2 \text{ et } \delta = -3$$

erit primo

$$lV = \int \frac{du}{2-4u+2uu} = \frac{+1}{2(1-u)} = \frac{x}{2(x-y)}$$

et quia non eſt $\delta = -\alpha$ ſolutio generalis ſtatim prodit:

$$z = (2xx-4xy+2yy)^{\frac{-1}{2}} f : \frac{(2xx-4xy+2yy)}{V^{-2}}$$

et ob

$$V = e^{\frac{x}{2(x-y)}} \text{ erit}$$

$$z = \frac{1}{x-y} f : (x-y)^2 e^{\frac{x}{x-y}}.$$

Vnde ſolutio ſimpliciſſima eſt $z = \frac{1}{x-y}$.

Scholion.

208. Hic merito quaerimus, quo pacto haec ſolutio generalis ſtatim ſine adiumento ſolutionis ſpecialis inueniri potuiſſet, ſequenti autem modo iſta inueſtigatio inſtituenda videtur. Cum ſit

$$p(\alpha x + \beta y) = z - q(\gamma x + \delta y) \text{ et}$$
$$q(\gamma x + \delta y) = z - p(\alpha x + \beta y)$$

ſi

si vterque valor seorsim in forma

$$dz = p\,dx + q\,dy$$

substituatur, prodibunt binae sequentes aequationes

$$(\alpha x+\beta y)dz = z\,dx - q(\gamma x+\delta y)dx + q(\alpha x+\beta y)dy$$

$$(\gamma x+\delta y)dz = z\,dy + p(\gamma x+\delta y)dx - p(\alpha x+\beta y)dy.$$

Multiplicetur prior indefinite per M posterior per N, et productorum summa dabit

$$dz(M(\alpha x+\beta y)+N(\gamma x+\delta y)) - z(M\,dx+N\,dy)$$
$$=(Np-Mq)((\gamma x+\delta y)dx-(\alpha x+\beta y)dy)$$

vbi iam M et N ita capi debent, vt prius membrum integrationem admittat, tum enim eius integrale aequabitur functioni cuicunque quantitatis

$$\int\frac{(\gamma x+\delta y)dx-(\alpha x+\beta y)dy}{\gamma xx+(\delta-\alpha)xy-\beta yy},$$

quam supra (§. 197.) definire docuimus: vnde patet illud integrale fieri $=f:\frac{z}{U}$. Manifestum autem est M et N eiusmodi functiones esse oportere vt haec aequatio fiat possibilis

$$\frac{dz}{z} = \frac{M\,dx+N\,dy}{M(\alpha x+\beta y)+N(\gamma x+\delta y)}$$

seu vt membrum posterius integrationem admittat; quod si enim eius integrale sit $=lV$ erit $\frac{z}{V}=f:\frac{z}{U}$. Pro hac integrabilitate ponamus $y=ux$, et M et N functiones ipsius u, erit

$$\frac{dz}{z} = \frac{(M+Nu)dx+Nx\,du}{Mx(\alpha+\beta u)+Nx(\gamma+\delta u)}.$$

Vbi

Vbi integratio ſuccedit ſumendo $M=-Nu$; vt ſit

$$\frac{dz}{z}=\frac{+du}{\gamma+(\delta-\alpha)u-\beta uu} \text{ ſeu}$$

$$lV=\int\frac{du}{\gamma+(\delta-\alpha)u-\beta uu},$$

prorſus vt ante.

Problema 36.

209. Si poſito $dz=pdx+qdy$ debeat eſſe $Z=pP+qQ$ exiſtente Z functione ipſius z tantum, P et Q autem functionibus ipſarum x et y quibusuis datis, indolem functionis z inueſtigare.

Solutio.

Formentur ſequentes aequationes ex propoſitis:

$$Ldz=Lpdx+Lqdy;\ MZdx=MpPdx+MqQdx;$$
$$NZdy=NpPdy+NqQdy$$

quae in vnam ſummam collectae dabunt:

$$Ldz+Z(Mdx+Ndy)=p((L+MP)dx+NPdy)$$
$$+q((L+NQ)dy+MQdx).$$

Vt iam pars poſterior habeat factorem a litteris p et q liberum, fiat

$$L+MP:\ NP=MQ:\ L+NQ$$

vnde fit

$$LL+LNQ+LMP=0 \text{ ſeu } L=-MP-NQ$$

quo valore inducto erit

$$-dz(MP+NQ)+Z(Mdx+Ndy)=(Mq-Np)(Qdx-Pdy).$$

Cum

Cum nunc P et Q sint functiones datae ipsarum x et y dabitur multiplicator R vt fiat

$$R(Qdx-Pdy)=dU,$$

ideoque

$$-dz(MP+NQ)+Z(Mdx+Ndy)=\frac{Mq-Np}{R}.dU$$

Pro parte priori capiantur functiones indefinitae M et N ita vt formula $\frac{Mdx+Ndy}{MP+NQ}$ integrabilis euadat id quod semper fieri licet. Sitque

$$\frac{Mdx+Ndy}{MP+NQ}=dV,$$

et ob

$$Mdx+Ndy=(MP+NQ)dV$$

aequatio nostra hanc induet formam:

$$(MP+NQ)(-dz+ZdV)=\frac{Mq-Np}{R}.dU \text{ seu}$$

$$\frac{dz}{Z}-dV=\frac{Np-Mq}{RZ(MP+NQ)}.dU.$$

Statuatur iam

$$\frac{Np-Mq}{RZ(MP+NQ)}=f':U,$$

atque habebitur:

$$\int\frac{dz}{Z}-V=f:U \text{ seu } \int\frac{dz}{Z}=V+f:U$$

vnde z determinatur per x et y.

Coroll. 1.

210. Pro solutione ergo problematis quaeratur primo ad formulam $Qdx-Pdy$ multiplicator R

eam reddens integrabilem statuaturque

$$R(Q\,dx - P\,dy) = dU,$$

vnde colligitur quantitas U per binas variabiles x et y expressa.

Coroll. 2.

211. Deinde quantitates M et N ita capiantur, vt formula $\frac{M\,dx + N\,dy}{MP + NQ}$ fiat integrabilis, cuius integrale si statuatur $= V$ statim habetur solutio generalis problematis, quae dat:

$$\int \frac{dz}{Z} = V + f : U.$$

Exemplum.

212. *Si* P *et* Q *sint functiones homogeneae ipsarum* x *et* y *vtraque dimensionum numeri* $= n$, *solutionem problematis perficere.*

Ponatur $y = ux$, et tam P quam Q fiet productum ex potestate x^n in functionem quandam ipsius u. Sit ergo

$$P = x^n S \quad \text{et} \quad Q = x^n T,$$

eruntque S et T functiones datae ipsius u. Tum vero ob $dy = u\,dx + x\,du$, formula $Q\,dx - P\,dy$ abit in

$$x^n T\,dx - x^n Su\,dx - x^{n+1} S\,du = x^n((T - Su)dx - Sx\,du).$$

Sumatur ergo

$$R = \frac{1}{x^{n+1}(T - Su)}, \text{ fietque}$$

$$dU = \frac{dx}{x} - \frac{S\,du}{T - Su}, \text{ vnde colligitur } U.$$

Deinde

Deinde pro altera quantitate V habebimus hanc aequationem

$$dV = \frac{(M+Nu)dx + Nx\,du}{x^n(MS+NT)},$$

vbi iam facile est pro M et N eiusmodi functiones ipsius u assumere vt haec formula integrationem admittat. Integrale scilicet erit

$$V = \frac{-M-Nu}{(n-1)x^{n-1}(MS+NT)}$$

at M et N seu $\frac{M}{N}=K$ ita accipi debet vt fiat

$$\frac{1}{(n-1)x^{n-1}}\,d.\,\frac{K+u}{KS+T} = \frac{1}{x^{n-1}}\cdot\frac{du}{KS+T} \text{ seu}$$

$$-KKdS+KSdu-uKdS-uSdK+TdK-KdT$$
$$+Tdu-udT+(n-1)du(KS+T)=0$$

quae ad hanc formam reducitur:

$$(T-Su)dK+K(nSdu-udS-dT)-KKdS$$
$$+nTdu-udT=0.$$

Ex qua, concessa aequationum resolutione cognoscitur quantitas K, qua inuenta erit

$$V = \frac{-K-u}{(n-1)x^{n-1}(KS+T)}.$$

Cum autem illa aequatio solutu difficilis videatur,

 pona-

ponatur statim $\frac{K+u}{KS+T}=v$, eritque

$$K=\frac{Tv-u}{1-Sv} \quad \text{et} \quad KS+T=\frac{T-Su}{1-Sv},$$

vnde fit

$$dv+\frac{(n-1)du(1-Sv)}{T-Su}=0,$$

qua resoluta erit $V=\frac{-v}{(n-1)x^{n-1}}$.

Corollarium.

213. Casus autem quo $n=1$ singulari euolutione eget. Facile autem patet tum sumi debere $M=-Nu$, vt fiat $dV=\frac{du}{T-Su}$ vnde postquam quantitas V fuerit inuenta, erit semper

$$\int\frac{dz}{Z}=V+f:U.$$

Scholion.

214. Cum ternae variabiles x, y, z sint inter se permutabiles patet hoc problema multo latius extendi posse. Scilicet si conditio proposita hac contineatur aequatione $pP+qQ+R=0$ non solum soluendi methodus adhibita succedit, si R sit functio ipsius z, et P cum Q functiones ipsarum x et y, sed etiam si fuerit P functio ipsius x et Q et R functiones ipsarum y et z. Tum vero etiam si Q functio ipsius y, at P et R functiones binarum reliquarum x et z. Haec vero conditio cum ante tractatis eo redit, vt binae formulae differentiales p et q sint a se inuicem separatae, neque plus vna dimensione occupent,

etiamsi

etiamsi et his casibus ingens restrictio accedat. Quodsi autem conditio magis sit complicata solutio vix vnquam sperari posse videtur, interim tamen casum eiusmodi proferam, quo solutionem expedire licet.

Problema 37.

215. Si posito $dz = pdx + qdy$ debeat esse

$$q = A p^n x^\lambda y^\mu z^\nu,$$

indolem functionis z in genere inuestigare.

Solutio.

Posito hoc valore loco q habebimus:

$$dz = pdx + Ap^n x^\lambda y^\mu z^\nu dy,$$

vnde fit

$$Ay^\mu dy = p^{-n} x^{-\lambda} z^{-\nu}(dz - pdx).$$

Ponatur $p^{-n} x^{-\lambda} z^{-\nu} = t$ vt sit

$$p = t^{-\frac{1}{n}} x^{-\frac{\lambda}{n}} z^{-\frac{\nu}{n}} \text{ eritque}$$

$$Ay^\mu dy = tdz - t^{\frac{n-1}{n}} x^{-\frac{\lambda}{n}} z^{-\frac{\nu}{n}} dx.$$

Statuatur porro

$$t^{n-1} z^{-\nu} = u^n \text{ seu } t = z^{\frac{\nu}{n-1}} u^{\frac{n}{n-1}} \text{ erit}$$

$$Ay^\mu dy = u^{\frac{n}{n-1}} z^{\frac{\nu}{n-1}} dz - ux^{-\frac{\lambda}{n}} dx.$$

posteriori autem

$$\frac{n}{n-\lambda}x^{\frac{n-\lambda}{n}}=lx,$$

quos valores in solutionem introduci oportet.

Exemplum.

218. *Si posito* $dz=pdx+qdy$ *debeat esse* $pqxy=az$, *seu* $q=\frac{az}{pxy}$, *functionem* z *inuestigare.*

Erit ergo

$$dz=pdx+\frac{azdy}{pxy} \text{ seu } \frac{ady}{y}=\frac{px}{z}(dz-pdx).$$

Ponatur

$$\frac{px}{z}=t \text{ seu } p=\frac{tz}{x} \text{ erit,}$$

$$\frac{ady}{y}=tdz-\frac{ttzdx}{x}.$$

Statuatur porro

$$ttz=uu \text{ seu } t=\frac{u}{\sqrt{z}}$$

vt fit

$$\frac{ady}{y}=\frac{udz}{\sqrt{z}}-\frac{uudx}{x},$$

et quoad fieri potest integrando:

$$aly=2u\sqrt{z}-uulx-\int du(2\sqrt{z}-2ulx)$$

ita vt iam post signum integrale vnicum differentiale du reperiatur. Posito ergo

$$\sqrt{z}-ulx=f':u \text{ erit}$$

$$aly=2u\sqrt{z}-uulx-2f:u=uulx+2uf':u-2f:u.$$

Pro

Pro casu simplicissimo sumatur $f':u=0$ et $f:u=0$ erit $u=\frac{\sqrt{z}}{lx}$ ideoque

$$aly=\frac{2z}{lx}-\frac{z}{lx}=\frac{z}{lx},$$

ita vt pro casu simplicissimo sit $z=alx.ly$.

Si ponatur

$$f':u=ulc \text{ et } f:u=\tfrac{1}{2}uulc \text{ erit}$$

$$u=\frac{\sqrt{z}}{lx+lc}=\frac{\sqrt{z}}{lcx} \text{ et}$$

$$aly=\frac{2z}{lcx}-\frac{zlx}{(lcx)^2}-\frac{zlc}{(lcx)^2}=\frac{lz}{lcx},$$

ita vt sit

$$z=aly(lc+lx),$$

magis generaliter autem erit

$$z=a(lb+ly)(lc+lx).$$

Scholion.

219. Methodi hactenus traditae haud mediocriter amplificabuntur, si loco binarum variabilium x et y, quarum functio esse debet z binae aliae variabiles t et u introducantur, quarum relatio ad illas detur. Ita si z sit functio binarum variabilium x et y, vt inde prodeat

$$dz=pdx+qdy;$$

ac loco x et y aliae nouae variabiles t et u introducantur, vt iam differentiatione instituta prodeat

$$dz=rdt+sdu;$$

quae-

quaeritur quomodo r et s per p et q determinentur, pro relatione inter priſtinas variabiles x, y et nouas t et u ſtabilita. Hinc ergo tam x quam y certae cuidam functioni ipſarum t et u aequabitur, quae cum detur ſit

$$dx = P\,dt + Q\,du \quad \text{et} \quad dy = R\,dt + S\,du,$$

ita vt facta hac ſubſtitutione z iam ſit functio ipſarum t et u. Cum igitur eſſet

$$dz = p\,dx + q\,dy$$

erit nunc

$$dz = (Pp + Rq)\,dt + (Qp + Sq)\,du.$$

Eſt vero per hypotheſin

$$dz = r\,dt + s\,du,$$

vnde habebitur

$$r = Pp + Rq \quad \text{et} \quad s = Qp + Sq.$$

Quare facta hac ſubſtitutione valores differentiales noui ex praecedentibus ita determinabuntur vt ſit

$$\left(\frac{dz}{dt}\right) = P\left(\frac{dz}{dx}\right) + R\left(\frac{dz}{dy}\right) \quad \text{et} \quad \left(\frac{dz}{du}\right) = Q\left(\frac{dz}{dx}\right) + S\left(\frac{dz}{dy}\right).$$

Vnde etiam cum ſit viciſſim

$$Qr - Ps = (QR - PS)q \quad \text{et} \quad Sr - Rs = (PS - QR)p,$$

concludimus fore

$$\left(\frac{dz}{dx}\right) = \frac{S}{PS - QR}\left(\frac{dz}{dt}\right) - \frac{R}{PS - QR}\left(\frac{dz}{du}\right) \quad \text{et}$$

$$\left(\frac{dz}{dy}\right) = \frac{-Q}{PS - QR}\left(\frac{dz}{dt}\right) + \frac{P}{PS - QR}\left(\frac{dz}{du}\right).$$

Vel cum x et y perinde ac z sint functiones ipsarum t et u haec relatio ita exprimi potest, vt sit

$$\left(\frac{dz}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right) \text{ et}$$

$$\left(\frac{dz}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Hinc efficitur, vt quae problemata pro data quadam relatione inter p, q, x, y, z resolui possunt, ea quoque pro relatione inde resultante inter r, s, [illegible] resolui queant; vnde saepe problemata nascuntur, quae solutu vehementer [illegible] tur; ex quo non contemnenda subsidia in hanc Analyseos partem inferri possent; sed quia vsus praecipue in formulis differentialibus secundi gradus spectatur, his non fusius immorans ad eas euoluendas progredior.

CALCVLI

CALCVLI INTEGRALIS
LIBER POSTERIOR.

PARS PRIMA

SEV

INVESTIGATIO FVNCTIONVM DVARVM VARIABILIVM EX DATA DIFFERENTIALIVM CVIVSVIS GRADVS RELATIONE.

SECTIO SECVNDA

INVESTIGATIO DVARVM VARIABILIVM FVNCTIONVM EX DATA DIFFERENTIALIVM SECVNDI GRADVS RELATIONE.

CAPVT I.

DE FORMVLIS DIFFERENTIALIBVS SECVNDI GRADVS IN GENERE.

Problema 38.

220.

Si z fit functio quaecunque binarum variabilium x et y, eius formulas differentiales fecundi gradus exhibere.

Solutio.

Cum z fit functio binarum variabilium x et y, eius differentiale huiusmodi habebit formam

$$dz = p\,dx + q\,dy,$$

ex qua p et q funt formulae differentiales primi

 gradus,

gradus, quas ita denotare solemus:

$$p = \left(\frac{dz}{dx}\right) \quad \text{et} \quad q = \left(\frac{dz}{dy}\right).$$

Cum nunc sint quoque p et q functiones ipsarum x et y, formulae differentiales inde natae erunt formulae differentiales secundi gradus ipsius z, vnde intelligitur quatuor huiusmodi formulas nasci:

$$\left(\frac{dp}{dx}\right); \left(\frac{dp}{dy}\right); \left(\frac{dq}{dx}\right); \left(\frac{dq}{dy}\right)$$

quarum autem secundam ac tertiam inter se congruere in Calculo differentiali est demonstratum. Sed cum sit $p = \left(\frac{dz}{dx}\right)$, simili scribendi ratione erit $\left(\frac{dp}{dx}\right) = \left(\frac{ddz}{dx^2}\right)$ cuius scripturae significatus hinc sponte patet. Deinde eodem modo erit $\left(\frac{dp}{dy}\right) = \left(\frac{ddz}{dxdy}\right)$; atque ob $q = \left(\frac{dz}{dy}\right)$ habebimus

$$\left(\frac{dq}{dx}\right) = \left(\frac{ddz}{dydx}\right) \text{ et } \left(\frac{dq}{dy}\right) = \left(\frac{ddz}{dy^2}\right).$$

Quia ergo est $\left(\frac{ddz}{dydx}\right) = \left(\frac{ddz}{dxdy}\right)$, functioni z conveniunt tres formulae differentiales secundi gradus, quae sunt:

$$\left(\frac{ddz}{dx^2}\right); \left(\frac{ddz}{dxdy}\right) \text{ et } \left(\frac{ddz}{dy^2}\right).$$

Coroll. 1.

221. Vt ergo functio z duarum variabilium x et y duas habet formulas differentiales primi gradus

$$\left(\frac{dz}{dx}\right) \text{ et } \left(\frac{dz}{dy}\right),$$

ita habet tres formulas differentiales secundi gradus

$$\left(\frac{ddz}{dx^2}\right); \left(\frac{ddz}{dxdy}\right) \text{ et } \left(\frac{ddz}{dy^2}\right).$$

Coroll. 2.

222. Hae ergo formulae per duplicem differentiationem naſcuntur, vnicam tantum quantitatem pro variabili accipienda. In prima ſcilicet bis eadem x variabilis ſumitur, in ſecunda vero in altera differentiatione x, in altera autem y variabilis accipitur; in tertia autem bis y.

Coroll. 3.

[illegible] Simili modo patet eiusdem functionis z quatuor dari formulas [illegible] ſcilicet:

$$\left(\frac{d^3 z}{dx^3}\right);\ \left(\frac{d^3 z}{dx^2 dy}\right);\ \left(\frac{d^3 z}{dx\, dy^2}\right);\ \left(\frac{d^3 z}{dy^3}\right)$$

quarti autem gradus quinque; quinti, ſex etc.

Scholion.

224. Formulae hae differentiales ſecundi gradus ope ſubſtitutionis ſaltem ad formam primi gradus reuocari poſſunt. Veluti formula $\left(\frac{ddz}{dx^2}\right)$, ſi ponatur $\left(\frac{dz}{dx}\right) = p$, transformabitur in $\left(\frac{dp}{dx}\right)$; formula autem $\left(\frac{ddz}{dx\,dy}\right)$ eadem ſubſtitutione in hanc $\left(\frac{dp}{dy}\right)$. At poſito $\left(\frac{dz}{dy}\right) = q$, formula $\left(\frac{ddz}{dx\,dy}\right)$ transmutatur in hanc $\left(\frac{dq}{dx}\right)$, formula autem $\left(\frac{ddz}{dy^2}\right)$ in hanc $\left(\frac{dq}{dy}\right)$. Viciſſim autem vti ex aequalitate $p = \left(\frac{dz}{dx}\right)$ deduximus

$$\left(\frac{dp}{dx}\right) = \left(\frac{ddz}{dx^2}\right) \text{ et } \left(\frac{dp}{dy}\right) = \left(\frac{ddz}{dx\,dy}\right),$$

ita

ita ex his vlterius progrediendo colligemus:

$$\left(\frac{ddp}{dx^2}\right)=\left(\frac{d^3z}{dx^3}\right);\ \left(\frac{ddp}{dxdy}\right)=\left(\frac{d^3z}{dx^2dy}\right);\ \left(\frac{ddp}{dy^2}\right)=\left(\frac{d^3z}{dxdy^2}\right).$$

Tum vero etiam si ponamus $\left(\frac{dp}{dx}\right)=\left(\frac{dz}{dy}\right)$, hinc sequentur istae aequalitates

$$\left(\frac{ddp}{dx^2}\right)=\left(\frac{ddz}{dxdy}\right) \text{ et } \left(\frac{ddp}{dxdy}\right)=\left(\frac{ddz}{dy^2}\right).$$

Hicque est quasi nouus algorithmus, cuius principia per se ita sunt manifesta, vt maiore illustratione non indigeant.

Exemplum 1.

225. *Si sit* $z=xy$ *eius formulas differentiales secundi gradus exhibere.*

Cum sit

$$\left(\frac{dz}{dx}\right)=y \text{ et } \left(\frac{dz}{dy}\right)=x \text{ erit}$$

$$\left(\frac{ddz}{dx^2}\right)=0;\ \left(\frac{ddz}{dxdy}\right)=1 \text{ et } \left(\frac{ddz}{dy^2}\right)=0.$$

Exemplum 2.

226. *Si sit* $z=x^m y^n$ *eius formulas differentiales secundi gradus exhibere.*

Cum sit

$$\left(\frac{dz}{dx}\right)=mx^{m-1}y^n \text{ et } \left(\frac{dz}{dy}\right)=nx^my^{n-1} \text{ erit}$$

$$\left(\frac{ddz}{dx^2}\right)=m(m-1)x^{m-2}y^n;\ \left(\frac{ddz}{dxdy}\right)=mnx^{m-1}y^{n-1};$$

$$\left(\frac{ddz}{dy^2}\right)=n(n-1)x^my^{n-2}.$$

Exem-

Exemplum 3.

227. *Si fit* $z = \sqrt{(xx+yy)}$, *eius formulas differentiales secundi gradus exhibere.*

Cum fit

$$\left(\frac{dz}{dx}\right) = \frac{x}{\sqrt{(xx+yy)}} \text{ et } \left(\frac{dz}{dy}\right) = \frac{y}{\sqrt{(xx+yy)}} \text{ erit}$$

$$\left(\frac{ddz}{dx^2}\right) = \frac{yy}{(xx+yy)^{\frac{3}{2}}}; \quad \left(\frac{ddz}{dxdy}\right) = \frac{-xy}{(xx+yy)^{\frac{3}{2}}};$$

$$\left(\frac{ddz}{dy^2}\right) = \frac{xx}{(xx+yy)^{\frac{3}{2}}}.$$

Scholion.

228. Quemadmodum binae formulae differentiales primi gradus cuiusque functionis z ita funt comparatae, vt fit

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right),$$

et integrando

$$z = \int\left(dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right)\right)$$

ita quoque in formulis fecundi gradus erit:

$$\left(\frac{dz}{dx}\right) = \int\left(dx\left(\frac{ddz}{dx^2}\right) + dy\left(\frac{ddz}{dxdy}\right)\right) \text{ et}$$

$$\left(\frac{dz}{dy}\right) = \int\left(dx\left(\frac{ddz}{dxdy}\right) + dy\left(\frac{ddz}{dy^2}\right)\right).$$

Tres igitur formulae fecundi gradus femper ita funt comparatae, vt geminam integrationem praebeant, si scilicet cum differentialibus dx et dy rite combinentur, haecque proprietas quae probe notetur, in sequentibus insigne adiumentum afferet.

Problema 39.

229. Si z sit functio binarum variabilium x et y, loco x et y introducantur binae nouae variabiles t et u, ita vt tam x quam y aequetur certae functioni ipsarum t et u, formulas differentiales secundi gradus ipsius z respectu harum nouarum variabilium definire.

Solutio.

Quatenus z per x et y datur, datae sunt eius formulae differentiales tam primi gradus $(\frac{dz}{dx})$, $(\frac{dz}{dy})$, quam secundi gradus $(\frac{ddz}{dx^2})$, $(\frac{ddz}{dxdy})$, $(\frac{ddz}{dy^2})$, ex quibus quomodo formulae differentiales respectu nouarum variabilium t et u determinentur definiri oportet. Pro primo gradu autem cum sit

$$dz = dx(\frac{dz}{dx}) + dy(\frac{dz}{dy}),$$

quia tam x quam y datur per t et u erit

$$dx = dt(\frac{dx}{dt}) + du(\frac{dx}{du}) \text{ et } dy = dt(\frac{dy}{dt}) + du(\frac{dy}{du})$$

quibus valoribus substitutis habebitur ipsius z differentiale plenum ex variatione vtriusque t et u ortum:

$$dz = dt(\frac{dx}{dt})(\frac{dz}{dx}) + du(\frac{dx}{du})(\frac{dz}{dx}) + dt(\frac{dy}{dt})(\frac{dz}{dy}) + du(\frac{dy}{du})(\frac{dz}{dy}).$$

Quodsi

Quodsi iam vel sola t variabilis sumatur, vel sola u, prodibunt formulae differentiales primi gradus:

$$\left(\frac{dz}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right);\quad \left(\frac{dz}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Simili modo vlterius progrediendo, differentiemus formulas

$$\left(\frac{dz}{dx}\right)=p \quad \text{et} \quad \left(\frac{dz}{dy}\right)=q$$

primo generaliter, tum vero loco x et y etiam t et u introducamus; hincque nanciscemur:

$$\left(\frac{dp}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dp}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dp}{dy}\right);\quad \left(\frac{dp}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dp}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dp}{dy}\right)$$

$$\left(\frac{dq}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dq}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dq}{dy}\right);\quad \left(\frac{dq}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dq}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dq}{dy}\right)$$

vnde poterimus formulas $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ pro variabilitate tam solius t quam solius u assignare; scilicet cum sit:

$$\left(\frac{dz}{dt}\right)=p\left(\frac{dx}{dt}\right)+q\left(\frac{dy}{dt}\right) \quad \text{et} \quad \left(\frac{dz}{du}\right)=p\left(\frac{dx}{du}\right)+q\left(\frac{dy}{du}\right)$$

inueniemus:

$$\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddx}{dt^2}\right)\left(\frac{dz}{dx}\right)+\left(\frac{ddy}{dt^2}\right)\left(\frac{dz}{dy}\right)+\left(\frac{dx}{dt}\right)^2\left(\frac{ddz}{dx^2}\right)$$
$$+2\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{dt}\right)^2\left(\frac{ddz}{dy^2}\right)$$

$$\left(\frac{ddz}{dtdu}\right)=\left(\frac{ddx}{dtdu}\right)\left(\frac{dz}{dx}\right)+\left(\frac{ddy}{dtdu}\right)\left(\frac{dz}{dy}\right)+\left(\frac{dx}{dt}\right)\left(\frac{dx}{du}\right)\left(\frac{ddz}{dx^2}\right)$$
$$+\left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dx}{du}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dy^2}\right)$$

$$\left(\frac{ddz}{du^2}\right)=\left(\frac{ddx}{du^2}\right)\left(\frac{dz}{dx}\right)+\left(\frac{ddy}{du^2}\right)\left(\frac{dz}{dy}\right)+\left(\frac{dx}{du}\right)^2\left(\frac{ddz}{dx^2}\right)$$
$$+2\left(\frac{dx}{du}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{du}\right)^2\left(\frac{ddz}{dy^2}\right).$$

Coroll. I.

Coroll. 1.

230 Proposita ergo conditione quadam inter formulas differentiales functionis z quatenus per variabiles t et u definitur, eadem conditio pro eadem functione z transfertur ad alias binas variabiles x et y, ab illis vtcunque pendentes.

Coroll. 2.

231. Formulae quidem

$$\left(\frac{dx}{dt}\right),\ \left(\frac{dy}{dt}\right),\ \left(\frac{dx}{du}\right),\ \left(\frac{dy}{du}\right) \text{ etc.}$$

per t et u exprimuntur, ex relatione, quae inter x, y et t, u assumitur verum indidem eaedem formulae ad variabiles x et y reuocari possunt.

Scholion.

232. Quemadmodum hic variabilitas quantitatum t et u per formulas differentiales ex variabilibus x et y natas est expressa, ita vicissim si variabiles t et u proponantur, ex quibus certo modo alterae x et y determinentur, sequentes reductiones habebuntur, facta tantum variabilium permutatione. Primo scilicet pro formulis primi gradus

$$\left(\frac{dz}{dx}\right)=\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)+\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right);\ \left(\frac{dz}{dy}\right)=\left(\frac{dt}{dy}\right)\left(\frac{dz}{dt}\right)+\left(\frac{du}{dy}\right)\left(\frac{dz}{du}\right)$$

Pro

Pro formulis autem differentialibus secundi gradus:

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddt}{dx^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dx^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dx}\right)^2\left(\frac{ddz}{dt^2}\right)$$
$$+2\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)^2\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dxdy}\right)=\left(\frac{ddt}{dxdy}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dxdy}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dx}\right)\left(\frac{dt}{dy}\right)\left(\frac{ddz}{dt^2}\right)$$
$$+\left(\frac{dt}{dx}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)\left(\frac{dt}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right)$$
$$+2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right)$$

vbi determinatio litterarum t et u per alteras x et y considerari debet. Quoniam scilicet in conditionibus praescriptis binis variabilibus x et y vti solemus, earum loco alias quascunque t et u introducendo, loco illarum formularum differentialium has nouas formas ad variabiles t et u relatas adhibere poterimus, vbi deinceps relatio inter variabiles x, y et t, u ita est constituenda, vt quaestio solutu facilior euadat. Pro variis igitur huiusmodi relationibus exempla euoluamus.

Exemplum I.

233. *Si inter variabiles* x, y *et* t, u *haec relatio constituatur, vt sit*

$$t=\alpha x+\beta y \quad \textit{et} \quad u=\gamma x+\delta y,$$

reductionem formularum differentialium exhibere.

 Cum

Cum sit

$$\left(\frac{dt}{dx}\right)=\alpha;\ \left(\frac{dt}{dy}\right)=\beta;\ \left(\frac{du}{dx}\right)=\gamma;\ \left(\frac{du}{dy}\right)=\delta;$$

hincque formulae pro secundo gradu euanescant; habebimus pro formulis primi gradus:

$$\left(\frac{dz}{dx}\right)=\alpha\left(\frac{dz}{dt}\right)+\gamma\left(\frac{dz}{du}\right);\ \left(\frac{dz}{dy}\right)=\beta\left(\frac{dz}{dt}\right)+\delta\left(\frac{dz}{du}\right)$$

pro formulis autem secundi gradus:

$$\left(\frac{ddz}{dx^2}\right)=\alpha\alpha\left(\frac{ddz}{dt^2}\right)+2\alpha\gamma\left(\frac{ddz}{dt\,du}\right)+\gamma\gamma\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dx\,dy}\right)=\alpha\beta\left(\frac{ddz}{dt^2}\right)+(\alpha\delta+\beta\gamma)\left(\frac{ddz}{dt\,du}\right)+\gamma\delta\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dy^2}\right)=\beta\beta\left(\frac{ddz}{dt^2}\right)+2\beta\delta\left(\frac{ddz}{dt\,du}\right)+\delta\delta\left(\frac{ddz}{du^2}\right).$$

Coroll. 1.

234. Si sumatur $t=x$ et $u=x+y$, erit $\alpha=1$, $\beta=0$, $\gamma=1$, et $\delta=1$, erit $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dt}\right)+\left(\frac{dz}{du}\right)$, $\left(\frac{dz}{dy}\right)=\left(\frac{dz}{du}\right)$ atque

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddz}{dt^2}\right)+2\left(\frac{ddz}{dt\,du}\right)+\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dx\,dy}\right)=\left(\frac{ddz}{dt\,du}\right)+\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{du^2}\right).$$

Coroll. 2.

235. Etsi ergo hic est $t=x$ tamen non est $\left(\frac{dz}{dt}\right)=\left(\frac{dz}{dx}\right)$ cuius rei ratio est, quod in forma $\left(\frac{dz}{dx}\right)$ quan-

quantitas y sumitur constans, in $\left(\frac{dz}{dt}\right)$ vero quantitas $u=x+y$, id quod in genere notasse iuuat, ne ex aequalitate $t=x$ ad aequalitatem formularum $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dt}\right)$ concludamus.

Exemplum 2.

236. *Si inter variabiles* t, u *et* x, y *haec relatio constituatur vt sit* $t=\alpha x^m$ *et* $u=\beta y^n$, *reductionem exhibere.*

Hic ergo erit

$$\left(\frac{dt}{dx}\right)=m\alpha x^{m-1};\ \left(\frac{dt}{dy}\right)=0;\ \left(\frac{ddt}{dx^2}\right)=m(m-1)\alpha x^{m-2};\ \left(\frac{du}{dx}\right)=0;$$

$$\left(\frac{du}{dy}\right)=n\beta y^{n-1};\ \left(\frac{ddu}{dy^2}\right)=n(n-1)\beta y^{n-2};$$

vnde obtinemus pro formulis primi gradus:

$$\left(\frac{dz}{dx}\right)=m\alpha x^{m-1}\left(\frac{dz}{dt}\right);\ \left(\frac{dz}{dy}\right)=n\beta y^{n-1}\left(\frac{dz}{du}\right)$$

pro formulis autem secundi gradus:

$$\left(\frac{ddz}{dx^2}\right)=m(m-1)\alpha x^{m-2}\left(\frac{dz}{dt}\right)+mm\alpha\alpha x^{2m-2}\left(\frac{ddz}{dt^2}\right)$$

$$\left(\frac{ddz}{dxdy}\right)=mn\alpha\beta x^{m-1}y^{n-1}\left(\frac{ddz}{dtdu}\right)$$

$$\left(\frac{ddz}{dy^2}\right)=n(n-1)\beta y^{n-2}\left(\frac{dz}{du}\right)+nn\beta\beta y^{2n-2}\left(\frac{ddz}{du^2}\right)$$

in quibus formulis iam loco x et y earum valores per t et u induci debent.

Exem-

Exemplum 3.

237. *Si inter variabiles* t, u *et* x, y *haec relatio constituatur vt sit* $x = t$ *et* $\frac{x}{y} = u$, *formularum differentialium reductionem exhibere.*

Cum sit $t = x$ et $u = \frac{x}{y}$ erit

$$\left(\frac{dt}{dx}\right) = 1;\ \left(\frac{dt}{dy}\right) = 0;$$

hincque formulae inuoluentes ddt euanescunt. Porro

$$\left(\frac{du}{dx}\right) = \frac{1}{y} = \frac{u}{t};\ \left(\frac{du}{dy}\right) = \frac{-x}{yy} = \frac{-uu}{t};\ \left(\frac{ddu}{dx^2}\right) = 0;$$
$$\left(\frac{ddu}{dxdy}\right) = \frac{-1}{yy} = \frac{-uu}{tt};\ \left(\frac{ddu}{dy^2}\right) = \frac{2x}{y^3} = \frac{2u^3}{tt};$$

vnde pro formulis primi gradus habebimus:

$$\left(\frac{dz}{dx}\right) = \left(\frac{dz}{dt}\right) + \frac{u}{t}\left(\frac{dz}{du}\right);\ \left(\frac{dz}{dy}\right) = \frac{-uu}{t}\left(\frac{dz}{du}\right)$$

pro formulis autem secundi gradus:

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddz}{dt^2}\right) + \frac{2u}{t}\left(\frac{ddz}{dtdu}\right) + \frac{uu}{tt}\left(\frac{ddz}{du^2}\right)$$
$$\left(\frac{ddz}{dxdy}\right) = \frac{-uu}{tt}\left(\frac{dz}{du}\right) - \frac{uu}{t}\left(\frac{ddz}{dtdu}\right) - \frac{u^3}{tt}\left(\frac{ddz}{du^2}\right)$$
$$\left(\frac{ddz}{dy^2}\right) = \frac{2u^3}{tt}\left(\frac{dz}{du}\right) + \frac{u^4}{tt}\left(\frac{ddz}{du^2}\right).$$

Exemplum 4.

238. *Si inter variabiles* t, u *et* x, y *haec relatio constituatur, vt sit* $t = e^x$ *et* $u = e^x y$, *seu* $x = lt$ *et* $y = \frac{u}{t}$, *reductionem formularum differentialium exhibere.*

Hic

Hic ergo eſt

$(\frac{dt}{dx}) = e^x = t$; $(\frac{dt}{dy}) = 0$; $(\frac{ddt}{dx^2}) = e^x = t$; $(\frac{ddt}{dxdy}) = 0$.

Deinde

$(\frac{du}{dx}) = e^x y = u$; $(\frac{du}{dy}) = e^x = t$;

tum vero

$(\frac{ddu}{dx^2}) = e^x y = u$; $(\frac{ddu}{dxdy}) = e^x = t$; $(\frac{ddu}{dy^2}) = 0$.

Quare pro formulis primi gradus habebimus:

$(\frac{dz}{dx}) = t(\frac{dz}{dt}) + u(\frac{dz}{du})$; $(\frac{dz}{dy}) = t(\frac{dz}{du})$.

Pro formulis autem ſecundi gradus:

$(\frac{ddz}{dx^2}) = t(\frac{dz}{dt}) + u(\frac{dz}{du}) + tt(\frac{ddz}{dt^2}) + 2tu(\frac{ddz}{dtdu}) + uu(\frac{ddz}{du^2})$

$(\frac{ddz}{dxdy}) = t(\frac{dz}{du}) + tt(\frac{ddz}{dtdu}) + tu(\frac{ddz}{du^2})$

$(\frac{ddz}{dy^2}) = tt(\frac{ddz}{du^2})$.

Scholion.

239. In formulis generalibus §. 231. datis aſſumſimus valores variabilium t et u per x et y expreſſos dari, et vniuerſa euolutione facta tum demum pro x et y variabiles t et u reſtitui. Commodius ergo videatur, ſi ſtatim variabilium x et y valores per t et u expreſſi habeantur, verum inde valores formularum $(\frac{dt}{dx})$, $(\frac{dt}{dy})$ etc. nimis complicate exprimerentur, quam vt eas in calculum introducere liceret. Scilicet ſi x et y per t et u dentur,

tur, inde fit

$$\left(\frac{dt}{dx}\right) = \frac{\left(\frac{dy}{du}\right)}{\left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right) - \left(\frac{dx}{du}\right)\left(\frac{dy}{dt}\right)}$$

ac formulae ſecundi gradus multo magis prodituræ ſunt perplexae. Quouis ergo caſu, quo huiusmodi, reductione vtendum videtur, coniectura potius quam certa ratione idoneam variabilium immutationem colligi conueniet. Alia vero etiam datur reductio ſaepe inſignem vtilitatem afferens, dum ipſius functionis z quaeſitae forma mutatur, veluti ſi ponatur $z = Vv$, denotante V functionem datam ipſarum x et y, ita vt iam v ſit functio quaeſita; quin etiam haec noua quaeſita v alio modo cum datis implicari poteſt.

Problema 40.

240. Propoſita functione z binarum variabilium x et y, ac poſita $z = Pv$, ita vt P ſit data quaedam functio ipſarum x et y, formulas differentiales ipſius z per formulas differentiales nouae functionis v exprimere.

Solutio.

Cum ſit $z = Pv$ ex regulis differentiandi traditis habebimus primo formulas differentiales primi gradus:

$$\left(\frac{dz}{dx}\right) = \left(\frac{dP}{dx}\right)v + P\left(\frac{dv}{dx}\right) \text{ et } \left(\frac{dz}{dy}\right) = \left(\frac{dP}{dy}\right)v + P\left(\frac{dv}{dy}\right).$$

Atque

Atque hinc deinceps formulae differentiales secundi ordinis ita prodibunt expressae:

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddP}{dx^2}\right)v+2\left(\frac{dP}{dx}\right)\left(\frac{dv}{dx}\right)+P\left(\frac{ddv}{dx^2}\right)$$

$$\left(\frac{ddz}{dxdy}\right)=\left(\frac{ddP}{dxdy}\right)v+\left(\frac{dP}{dx}\right)\left(\frac{dv}{dy}\right)+\left(\frac{dP}{dy}\right)\left(\frac{dv}{dx}\right)+P\left(\frac{ddv}{dxdy}\right)$$

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddP}{dy^2}\right)v+2\left(\frac{dP}{dy}\right)\left(\frac{dv}{dy}\right)+P\left(\frac{ddv}{dy^2}\right)$$

vbi cum P sit functio data ipsarum x et y, eius formulae differentiales simul habentur.

Coroll. 1.

241. Si P esset functio ipsius x tantum, puta X tum posito $z=Xv$ erit

$$\left(\frac{dz}{dx}\right)=\frac{dX}{dx}v+X\left(\frac{dv}{dx}\right) \text{ et } \left(\frac{dz}{dy}\right)=X\left(\frac{dv}{dy}\right) \text{ tum}$$

$$\left(\frac{ddz}{dx^2}\right)=\frac{ddX}{dx^2}v+\frac{2dX}{dx}\left(\frac{dv}{dx}\right)+X\left(\frac{ddv}{dx^2}\right)$$

$$\left(\frac{ddz}{dxdy}\right)=\frac{dX}{dx}\left(\frac{dv}{dy}\right)+X\left(\frac{ddv}{dxdy}\right)$$

$$\left(\frac{ddz}{dy^2}\right)=X\left(\frac{ddv}{dy^2}\right).$$

Coroll. 2.

242. Transformatio haec easdem variabiles x et y seruat et tantum loco functionis z alia v introducitur; cum ante manente eadem functione z, binae variabiles x et y ad alias t et u sint reductae. Ex quo hae duae transformationes genere sunt diuersae.

Scholion 1.

243. Casus simplicior fuisset, si per additionem posuissemus $z = P + v$, vt esset P functio quaedam data ipsarum x et y; verum tum transformatio ita fit obuia, vt inuestigatione non indigeat: est enim manifesto

$$\left(\frac{dz}{dx}\right) = \left(\frac{dP}{dx}\right) + \left(\frac{dv}{dx}\right);\ \left(\frac{dz}{dy}\right) = \left(\frac{dP}{dy}\right) + \left(\frac{dv}{dy}\right)$$

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddP}{dx^2}\right) + \left(\frac{ddv}{dx^2}\right)$$

$$\left(\frac{ddz}{dxdy}\right) = \left(\frac{ddP}{dxdy}\right) + \left(\frac{ddv}{dxdy}\right)$$

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddP}{dy^2}\right) + \left(\frac{ddv}{dy^2}\right).$$

Neque vero etiam formas magis compositas euolui necesse est, veluti, si ponamus $z = \sqrt{(PP + vv)}$, quandoquidem talis forma vix vnquam vsum foret habitura.

Scholion 2.

244. Praemissis his principiis et transformationibus, negotium aggrediamur, et methodos aperiamus, ex data relatione inter formulas differentiales secundi gradus, et primi gradus, itemque ipsas quantitates principales, harum ipsarum relationem inuestigandi. Hic scilicet praeter ipsas quantitates x, y, et z, earumque formulas differentiales primi gradus $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ considerandae veniunt tres formulae differentiales secundi gradus $\left(\frac{ddz}{dx^2}\right)$; $\left(\frac{ddz}{dxdy}\right)$ et $\left(\frac{ddz}{dy^2}\right)$; quarum vel vna vel binae, vel omnes tres in relationem propositam ingredi possunt, vbi

in

insuper ingens discrimen formulae primi gradus, siue in relationem ingrediantur, siue secus, constituunt. Non solum autem nimis longum foret omnes combinationes vti in praecedente sectione fecimus, prosequi, sed etiam defectus idonearum methodorum impedit, quo minus singula quaestionum huc pertinentium genera percurramus. Capita igitur pertractanda ita instituamus, prout methodus soluendi patietur, ea, vbi nihil praestare licet penitus praetermissuri.

CAPVT II.

DE VNA FORMVLA DIFFERENTIALI SECVNDI GRADVS PER RELIQVAS QVANTITATES VTCVNQVE DATA.

Problema 41.

245.

Si z debeat esse eiusmodi functio ipsarum x et y vt formula secundi gradus $(\frac{ddz}{dx^2})$, aequetur functioni datae ipsarum x et y; indolem functionis z inuestigare.

Solutio.

Sit P functio ista data ipsarum x et y, ita vt esse debeat $(\frac{ddz}{dx^2}) = P$. Sumatur iam y constans, et cum sit

$$d.(\frac{dz}{dx}) = dx(\frac{ddz}{dx^2}) \text{ erit } d.(\frac{dz}{dx}) = P\,dx,$$

vnde integrando prodit

$$(\frac{dz}{dx}) = \int P\,dx + \text{Const.}$$

vbi in integratione $\int P\,dx$ quantitas y pro constante habetur, et constans adiicienda functionem quamcunque

cunque ipsius y denotabit, ita vt haec prima integratio praebeat

$$\left(\frac{dz}{dx}\right) = \int P\, dx + f:y.$$

Nunc iterum quantitate y vt constante spectata erit

$$dz = dx\left(\frac{dz}{dx}\right) \text{ seu } dz = dx\int P dx + dx f:y$$

vbi cum $\int P dx$ sit functio ipsarum x et y, quarum haec y constans assumitur, integratio denuo instituta dabit:

$$z = \int dx \int P dx + x f:y + F:y$$

quod est integrale completum aequationis differentio-differentialis propositae, $\left(\frac{ddz}{dx^2}\right) = P$; propterea quod duas functiones arbitrarias $f:y$ et $F:y$ complectitur, quarum vtramque ita pro lubitu accipere licet, vt etiam functiones discontinuae non excludantur.

Coroll. 1.

246. Quodsi ergo proponatur haec conditio $\left(\frac{ddz}{dx^2}\right) = 0$ eius integratio completa dabit

$$z = x f:y + F:y$$

ob $P = 0$, cuius veritas ex differentiatione perspicitur, vnde fit primo $\left(\frac{dz}{dx}\right) = f:y$, tum vero $\left(\frac{ddz}{dx^2}\right) = 0$.

Coroll. 2.

247. Eodem modo in genere integrale inuentum per differentiationem comprobatur. Cum enim inuenerimus

$$z = \int dx \int P\,dx + xf:y + F:y,$$

prima differentiatio praebet

$$\left(\frac{dz}{dx}\right) = \int P\,dx + f:y,$$

repetita vero $\left(\frac{ddz}{dx^2}\right) = P$.

Coroll. 3.

248. Simili modo si haec proponatur conditio $\left(\frac{ddz}{dy^2}\right) = Q$ existente Q functione quacunque ipsarum x et y, integrale completum reperitur:

$$z = \int dy \int Q\,dy + yf:x + F:x$$

vbi in geminato integrali $\int dy \int Q\,dy$ quantitas x pro constante habetur.

Scholion.

249. Hinc ratio integralium completorum, quae ex formulis differentialibus secundi gradus nascuntur, in genere perspicitur quae in hoc est sita, vt duae functiones arbitrariae inuehantur, vbi iterum notandum est; has functiones tam discontinuas quam continuas esse posse. Nisi ergo per totam hanc sectionem integralia duas huiusmodi functiones arbitrarias inuoluant, ea pro completis haberi nequeunt.

queant. Quotiescunque enim problema ad huiusmodi aequationem $\left(\frac{ddz}{dx^2}\right) = P$ perducit, eius indoles ſemper ita eſt comparata, vt tributo ipſi x certo quodam valore $x = a$, tam formula $\left(\frac{dz}{dx}\right)$ quam ipſa quantitas z datae cuipiam functioni ipſius y aequari poſſit. Quare ſi tam integrale $\int P dx$ quam hoc $\int dx \int P dx$ ita accipiatur, vt poſito $x = a$ euaneſcat, erit pro eodem caſu $x = a$, valor

$$\left(\frac{dz}{dx}\right) = f:y \text{ et } z = af:y + F:y,$$

vnde ex problematis natura vtraque functio $f:y$ et $F:y$ definitur. Haec autem applicatio ad omnes caſus fieri non poſſet, niſi integrale completum haberetur; quamobrem in hoc praecipue eſt incumbendum, vt omnium huiusmodi problematum integralia completa habeantur. Ceterum hic in perpetuum monendum duco, quoties huiusmodi formula integralis $\int P dx$ occurrit, ſemper ſolam quantitatem x variabilem accipi eſſe intelligendam; ſiquidem ſi etiam y variabilis acciperetur, formula $\int P dx$ ne ſignificatum quidem admitteret. Simili modo in formula $\int dx \int P dx$ intelligi debet, in vtraque integratione ſolam x variabilem aſſumi. Sin autem talis forma $\int dy \int P dx$ occurrat, intelligendum eſt integrale $\int P dx$ ex variabilitate ſolius x colligi debere, quod ſi ponatur $= R$, vt habeatur $\int R dy$, hic iam ſola y pro variabili erit habenda.

Exemplum 1.

250. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z *vt sit* $\left(\frac{ddz}{dx^2}\right)=\frac{xy}{a}$.

Cum hic sit $P=\frac{xy}{a}$, erit

$$\int P dx=\frac{xxy}{2a} \text{ et } \int dx \int P dx=\frac{x^3 y}{6a};$$

sicque habebitur ex prima integratione:

$$\left(\frac{dz}{dx}\right)=\frac{xxy}{2a}+f:y,$$

ita vt posito $x=a$, formula $\left(\frac{dz}{dx}\right)$ functioni cuicunque ipsius y aequari possit, seu applicatae curuae cuiuscunque, respondenti abscissae y. Tum vero altera integratione instituta, erit

$$z=\frac{x^3 y}{6a}+x f:y+F:y,$$

qui valor casu $x=a$ denuo functioni cuicunque ipsius y aequari potest.

Exemplum 2.

251. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z *vt sit* $\left(\frac{ddz}{dx^2}\right)=\frac{ax}{\sqrt{(xx+yy)}}$.

Ob $P=\frac{ax}{\sqrt{(xx+yy)}}$ erit

$\int P dx=a\sqrt{(xx+yy)}$, et

$$\int dx \int P dx=a\int dx\sqrt{(xx+yy)}=\tfrac{1}{2}ax\sqrt{(xx+yy)} + \tfrac{1}{2}ayy\, l\left(x+\sqrt{(xx+yy)}\right)$$

vnde

vnde prima integratio praebet:

$\left(\frac{dz}{dx}\right) = a\sqrt{(xx+yy)} + f:y$ altera vero

$z = \frac{1}{2}ax\sqrt{(xx+yy)} + \frac{1}{2}ayy\,l(x+\sqrt{(xx+yy)}) + xf:y + F:y.$

Exemplum 3.

252. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z, *vt sit* $\left(\frac{ddz}{dx^2}\right) = \frac{1}{\sqrt{(aa-xx-yy)}}$.

Cum sit $P = \frac{1}{\sqrt{(aa-xx-yy)}}$ erit

$\int P\,dx = \text{Ang. fin.} \frac{x}{\sqrt{(aa-yy)}}$

tum vero

$\int dx \int P\,dx = x\,\text{Ang. fin.} \frac{x}{\sqrt{(aa-yy)}} - \int \frac{x\,dx}{\sqrt{(aa-xx-yy)}}$.

Quare integratio prima praebet:

$\left(\frac{dz}{dx}\right) = \text{Ang. fin.} \frac{x}{\sqrt{(aa-yy)}} + f:y$

hincque ipſa functio quaeſita erit

$z = x\,\text{Ang. fin.} \frac{x}{\sqrt{(aa-yy)}} + \sqrt{(aa-xx-yy)} + xf:y + F:y.$

Exemplum 4.

253. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z, *vt sit* $\left(\frac{ddz}{dx^2}\right) = x\,\text{fin.}(x+y)$.

Ob $P = x\,\text{fin.}(x+y)$, erit

$\int P\,dx = \int x\,dx\,\text{fin.}(x+y) = -x\,\text{cof.}(x+y) + \int dx\,\text{cof.}(x+y)$

$= -x\,\text{cof.}(x+y) + \text{fin.}(x+y).$

Tum vero eſt:

$$\int x\,dx\,\text{cof.}(x+y)=x\,\text{fin.}(x+y)+\text{cof.}(x+y)$$

ideoque

$$\int dx\int P\,dx=-2\,\text{cof.}(x+y)-x\,\text{fin.}(x+y).$$

Quocirca ambo noſtra integralia erunt

$$\left(\tfrac{dz}{dx}\right)=\text{fin.}(x+y)-x\,\text{cof.}(x+y)+f:y \text{ et}$$

$$z=-2\,\text{cof.}(x+y)-x\,\text{fin.}(x+y)+xf:y+F:y.$$

Problema 42.

254. Si z debeat eſſe eiusmodi functio variabilium x et y, vt ſit

$$\left(\tfrac{ddz}{dx^2}\right)=P\left(\tfrac{dz}{dx}\right)+Q,$$

exiſtentibus P et Q functionibus quibusuis ipſarum x et y, indolem functionis z in genere inueſtigare.

Solutio.

Ponamus hic $\left(\tfrac{dz}{dx}\right)=v$, vt ſit $\left(\tfrac{ddz}{dx^2}\right)=\left(\tfrac{dv}{dx}\right)$, erit noſtra aequatio integranda:

$$\left(\tfrac{dv}{dx}\right)=Pv+Q$$

ſpectetur ergo ſola x vt variabilis, et ob $dv=dx\left(\tfrac{dv}{dx}\right)$ erit

$$dv=Pv\,dx+Q\,dx;$$

quae

quae per $e^{-\int P dx}$ multiplicata et integrata dat:

$$e^{-\int P dx} v = \int e^{-\int P dx} Q dx + f:y$$

ideoque

$$\left(\frac{dz}{dx}\right) = e^{\int P dx} \int e^{-\int P dx} Q dx + e^{\int P dx} f:y.$$

Retineatur sola x variabilis, spectata y vt constante et ob $dz = dx\left(\frac{dz}{dx}\right)$ erit

$$z = \int e^{\int P dx} dx \int e^{-\int P dx} Q dx + f:y \int e^{\int P dx} dx + F:y$$

quod ob binas functiones arbitrarias $f:y$ et $F:y$ est integrale completum.

Coroll. 1.

255. Problema hoc multo latius patet praecedente, cum conditio proposita etiam formulam primi gradus $\left(\frac{dz}{dx}\right)$ inuoluat, nihilo vero minus solutio feliciter successit.

Coroll. 2.

256. Hic ergo quadruplici integratione est opus, primo scilicet quaeri debet integrale $\int P dx$, quod si ponatur $= l R$ quaeri porro debet integrale

$$\int e^{\int P dx} dx = \int R dx,$$

quodsi ponamus $= S$, restat integrale

$$\int R dx \int \frac{Q dx}{R} = \int d S \int \frac{Q dx}{R},$$

 quod

quod abit in

$$S\int\frac{Q\,dx}{R}-\int\frac{QS\,dx}{R},$$

ita vt insuper hae duae formae integrari debeant.

Coroll. 3.

257. Eodem omnino modo resoluitur problema, quo esse debet

$$\left(\frac{ddz}{dy^2}\right)=P\left(\frac{dz}{dy}\right)+Q,$$

si P et Q fuerint functiones quaecunque datae ipsarum x et y. Reperitur enim:

$$\left(\frac{dz}{dy}\right)=e^{\int P\,dy}\int e^{-\int P\,dy}Q\,dy+e^{\int P\,dy}.f:x \text{ et}$$

$$z=\int e^{\int P\,dy}dy\int e^{-\int P\,dy}Q\,dy+f:x.\int e^{\int P\,dy}dy+F:x.$$

Exemplum 1.

258. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z, *vt sit* $\left(\frac{ddz}{dx^2}\right)=\frac{n}{x}\left(\frac{dz}{dx}\right)$.

Posito $\left(\frac{dz}{dx}\right)=v$, sumtoque solo x variabili, erit $\frac{dv}{dx}=\frac{nv}{x}$, ideoque $\frac{dv}{v}=\frac{n\,dx}{x}$, cuius integrale dat

$$v=\left(\frac{dz}{dx}\right)=x^n f:y.$$

Iam iterum sola x pro variabili habita, erit

$$dz=x^n\,dx\,f:y$$

cuius integrale completum est:

$$z=\frac{1}{n+1}x^{n+1}f:y+F:y.$$

Casu

Casu autem $n = -1$ seu $\left(\frac{ddz}{dx^2}\right) = \frac{-1}{x}\left(\frac{dz}{dx}\right)$ erit

$$\left(\frac{dz}{dx}\right) = \frac{1}{x} f:y \text{ et } z = lx.f:y + F:y.$$

Exemplum 2.

259. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z, *vt sit* $\left(\frac{ddz}{dx^2}\right) = \frac{n}{x}\left(\frac{dz}{dx}\right) + \frac{a}{xy}$.

Posito $\left(\frac{dz}{dx}\right) = v$, sumtoque solo x variabili erit

$$dv = \frac{nv\,dx}{x} + \frac{a\,dx}{xy}$$

quae aequatio per x^n diuisa et integrata praebet

$$\frac{v}{x^n} = \frac{a}{y}\int \frac{dx}{x^{n+1}} = \frac{-a}{n\,x^n\,y} + f:y \text{ seu}$$

$$v = \left(\frac{dz}{dx}\right) = \frac{-a}{ny} + x^n f:y.$$

Sit iterum sola x variabilis vt habeatur

$$dz = \frac{-a\,dx}{ny} + x^n\,dx\,f:y,$$

prodibitque integrale completum

$$z = \frac{-ax}{ny} + \frac{1}{n+1} x^{n+1} f:y + F:y.$$

Exemplum 3.

260. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z, *vt sit* $\left(\frac{ddz}{dx^2}\right) = \frac{2nx}{xx+yy}\left(\frac{dz}{dx}\right) + \frac{z}{ay}$.

Posito

Posito $\left(\frac{dz}{dx}\right) = v$, erit sumendo y constans:

$$dv = \frac{2nxv\,dx}{xx+yy} + \frac{x\,dx}{ay}$$

quae aequatio per $(xx+yy)^n$ diuisa et integrata dat:

$$\frac{v}{(xx+yy)^n} = \frac{1}{ay}\int\frac{xdx}{(xx+yy)^n} = -\frac{1}{2(n-1)ay(xx+yy)^{n-1}} + f:y$$

seu $v = \left(\frac{dz}{dx}\right) = \frac{-(xx+yy)}{2(n-1)ay} + (xx+yy)^n f:y.$

Hinc sumto iterum y constante fit

$$z = \frac{-x(xx+3yy)}{6(n-1)ay} + f:y.\int(xx+yy)^n dx + F:y.$$

Casu quo $n = 1$ seu

$$\left(\frac{ddz}{dx^2}\right) = \frac{2x}{xx+yy}\left(\frac{dz}{dx}\right) + \frac{x}{ay} \text{ erit}$$

$$\frac{v}{xx+yy} = \frac{1}{ay}\int\frac{xdx}{xx+yy} = \frac{1}{2ay}l((xx+yy) + f:y \text{ hinc}$$

$$\left(\frac{dz}{dx}\right) = \frac{xx+yy}{2ay}l(xx+yy) + (xx+yy)f:y \text{ et}$$

$$z = \frac{x(xx+3yy)}{6ay}l(xx+yy) - \frac{1}{9ay}(x^3 + 6xy^2 - 6y^3 \text{Ang. tang.} \tfrac{x}{y})$$
$$+ \tfrac{1}{3}x(xx+3yy)f:y + F:y.$$

Problema 43.

261. Si z debeat esse eiusmodi functio binarum variabilium x et y, vt sit

$$\left(\frac{ddz}{dx^2}\right) = P\left(\frac{dz}{dx}\right) + Q,$$

existentibus P et Q functionibus quibuscunque datis omnium trium variabilium x, y et z, indolem functionis z inuestigare.

Solutio.

Solutio.

Posita quantitate y constante, erit

$$\left(\frac{ddz}{dx^2}\right)=\frac{ddz}{dx^2} \text{ et } \left(\frac{dz}{dx}\right)=\frac{dz}{dx};$$

ideoque habebitur aequatio differentialis secundi gradus ad librum praecedentem pertinens:

$$ddz=Pdxdz+Qdx^2$$

quae duas tantum variabiles x et z inuoluere est censenda, quia y in ea tanquam constans spectatur. Tentetur ergo integratio huius aequationis per methodos ibi expositas; quae si successerit loco binarum constantium, quas duplex integratio inuehit scribantur ipsius y functiones indefinitae $f:y$ et $F:y$, quae adeo discontinuae accipi possunt, sicque habebitur aequationis propositae integrale completum.

Coroll. 1

162. Reducitur ergo solutio huius problematis ad methodum integrandi in superiori libro traditam, vbi functionem vnius variabilis ex data differentialium secundi gradus relatione inuestigari oportebat.

Coroll. 2.

263. Quodsi ergo resolutionem omnium aequationum differentialium secundi gradus, quae binas tantum variabiles inuoluunt, hic nobis concedi postulemus, solutio nostri problematis pro confecta est censenda.

Coroll. 3.

264. Me non monente intelligitur, eodem modo aequationem

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{dz}{dy}\right) + Q$$

tractari oportere, eiusque solutionem tanquam confectam spectari posse, quaecunque fuerint P et Q functiones ipsarum x, y et z.

Scholion 1.

265. Ex solutionis ratione intelligitur problema multo latius patens simili modo resolui posse: si enim formula $\left(\frac{ddz}{dx^2}\right)$ quomodocunque per quantitates principales x, y z ac praeterea formulam $\left(\frac{dz}{dx}\right)$ determinetur, ita vt etiam huius formulae $\left(\frac{dz}{dx}\right)$ potestates aliaeue functiones quaecunque ingrediantur, solutio semper ad librum superiorem reuocabitur; quia ponendo y constans fit

$$\left(\frac{dz}{dx}\right) = \frac{dz}{dx} \text{ et } \left(\frac{ddz}{dx^2}\right) = \frac{ddz}{dx^2},$$

ideoque resultat aequatio differentialis secundi gradus formae consuetae duas tantum variabiles x et z inuoluens. Hoc tantum teneatur loco constantium per vtramque integrationem ingredientium scribi oportere formas $f:y$ et $F:y$. Satis igitur notabilem partem propositi nostri expediuimus, scilicet cum vel $\left(\frac{ddz}{dx^2}\right)$ vtcunque per x, y, z et $\left(\frac{dz}{dx}\right)$, vel $\left(\frac{ddz}{dy^2}\right)$ vtcunque per x, y, z et $\left(\frac{dz}{dy}\right)$ determinatur, ibi nempe excluditur formula primi gradus $\left(\frac{dz}{dy}\right)$ hic vero

vero formula $\left(\frac{dz}{dx}\right)$. Quae si accederet, quaestio hac methodo neutiquam tractari posset; quemadmodum vel ex hoc casu simplicissimo $\left(\frac{ddz}{dx^2}\right)=\left(\frac{dz}{dy}\right)$ intelligere licet, cuius resolutio maxime ardua est putanda.

Scholion 2.

266. Cum igitur trium formularum differentialium secundi gradus $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$, $\left(\frac{ddz}{dy^2}\right)$ primam ac tertiam hactenus sim contemplatus, quatenus earum per reliquas quantitates determinatio resolutionem admittit methodo quidem hic adhibita: superest vt formulam quoque secundam $\left(\frac{ddz}{dxdy}\right)$ consideremus, et quibusnam determinationibus per reliquas quantitates x, y, z, $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ solutio absolui queat, inuestigemus, in quo negotio a casibus simplicissimis exordiri conueniet.

Problema 44.

267. Si z eiusmodi debeat esse functio binarum variabilium x et y vt fiat $\left(\frac{ddz}{dxdy}\right)=P$, existente P functione quacunque data ipsarum x et y, indolem functionis z generaliter determinare.

Solutio.

Ponatur $\left(\frac{dz}{dx}\right)=v$, eritque $\left(\frac{ddz}{dxdy}\right)=\left(\frac{dv}{dy}\right)$, ideoque habebitur $\left(\frac{dv}{dy}\right)=P$. Iam spectetur quantitas x

vt constans, ita vt P solam variabilem y contineat, eritque $dv = P dy$, vnde in hypothesi quantitatis x constantis integrando prodit

$$v = \left(\frac{dz}{dx}\right) = \int P dy + f' : x$$

vbi $\int P dy$ erit functio data ipsarum x et y. Nunc porro spectetur x vt variabilis, y vero vt constans, vt adipiscamur hanc aequationem differentialem:

$$dz = dx \int P dy + dx f' : x$$

quae integrata dat:

$$z = \int dx \int P dy + f : x + F : y$$

vbi cum habeantur duae functiones arbitrariae, id indicio est, hoc integrale esse completum.

Coroll. 1.

268. Si ordine inuerso primum y tum vero x constans posuissemus, inuenissemus

$$\left(\frac{dz}{dy}\right) = \int P dx + f' : y \text{ et } z = \int dy \int P dx + f : y + F : x,$$

qui valor aeque satisfacit ac praecedens.

Coroll. 2.

269. Patet ergo vel fore

$$\int dx \int P dy = \int dy \int P dx,$$

vel differentiam saltem exprimi per aggregatum ex functione ipsius x et functione ipsius y. Quod etiam inde patet quod posito

$$\int dx \int P dy = \int dy \int P dx = V$$

fiat vtrinque $P = \left(\frac{ddV}{dxdy}\right)$.

Coroll. 3.

270. Si sit $P = 0$, seu debeat esse $\left(\frac{ddz}{dxdy}\right) = 0$ reperitur pro indole functionis z haec forma

$$z = f : x + F : y.$$

Scholion.

271. Hic casus in doctrina solidorum frequenter occurrit, si enim natura superficiei exprimatur aequatione inter ternas coordinatas x, y et u, erit soliditas $= \int dx \int u dy$, quare si soliditas exprimatur per z, erit $\left(\frac{ddz}{dxdy}\right) = u$, ordinatae scilicet ad binas x et y normali. Tum vero si ponatur

$$du = p\,dx + q\,dy,$$

superficies huius solidi erit

$$= \int dx \int dy \sqrt{(1 + pp + qq)},$$

quae superficies si exprimatur littera z, erit

$$\left(\frac{ddz}{dxdy}\right) = \sqrt{(1 + pp + qq)}.$$

Quando ergo in nostro problemate eiusmodi functio z ipsarum x et y quaeritur, vt sit $\left(\frac{ddz}{dxdy}\right) = P$, idem est ac si quaeratur soliditas respondens superficiei, cuius natura aequatione inter ternas coordinatas

natas x, y et P exprimitur. Exemplis igitur aliquot hunc calculum illustremus.

Exemplum 1.

272. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z, *vt sit* $\left(\frac{ddz}{dxdy}\right)=\alpha x+\beta y$.

Cum hic sit $P=\alpha x+\beta y$ erit

$\int P dy=\alpha xy+\frac{1}{2}\beta yy$ et

$\int dx\int P dy=\frac{1}{2}\alpha xxy+\frac{1}{2}\beta xyy=\frac{1}{2}xy(\alpha x+\beta y)$,

vnde functio quaesita z ita exprimitur vt sit

$$z=\frac{1}{2}xy(\alpha x+\beta y)+f:x+F:y.$$

Exemplum 2.

273. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z, *vt sit* $\left(\frac{ddz}{dxdy}\right)=\sqrt{(aa-yy)}$.

Hic est $P=\sqrt{(aa-yy)}$, ergo

$$\int P dx=x\sqrt{(aa-yy)},$$

vbi quia perinde est, a variabilitate ipsius x incipio. Hinc igitur fit

$$\int dy\int P dx=x\int dy\sqrt{(aa-yy)}=\frac{1}{2}xy\sqrt{(aa-yy)}+\frac{1}{2}aax\int\frac{dy}{\sqrt{(aa-yy)}}$$

ex quo integrale completum erit

$$z=\frac{1}{2}xy\sqrt{(aa-yy)}+\frac{1}{2}aax\,\text{Ang. sin.}\frac{y}{a}+f:x+F:y.$$

Exem-

Exemplum 3.

274. *Quaeratur binarum variabilium* x *et* y *eiusmodi functio* z, *vt sit* $\left(\frac{ddz}{dxdy}\right) = \frac{a}{\sqrt{(aa - xx - yy)}}$.

Ob $P = \frac{a}{\sqrt{(aa - xx - yy)}}$ erit

$\int P dy = a \text{ Ang. fin. } \frac{y}{\sqrt{(aa - xx)}}$ hinc

$\int dx \int P dy = a \int dx \text{ Ang. fin. } \frac{y}{\sqrt{(aa - xx)}}$.

Ponatur breuitatis gratia

$$\text{Ang. fin. } \frac{y}{\sqrt{(aa - xx)}} = \Phi, \text{ erit}$$

$$\int dx \int P dy = a \int \Phi dx = ax\Phi - a \int x dx \left(\frac{d\Phi}{dx}\right)$$

in hac enim integratione y pro constante habetur. Quare ob $\frac{y}{\sqrt{(aa - xx)}} = \text{fin. } \Phi$, erit

$$\frac{yx}{(aa - xx)^{\frac{3}{2}}} = \left(\frac{d\Phi}{dx}\right) \text{cof. } \Phi.$$

At vero est

$$\text{cof. } \Phi = \frac{\sqrt{(aa - xx - yy)}}{\sqrt{(aa - xx)}}, \text{ hincque}$$

$$\left(\frac{d\Phi}{dx}\right) = \frac{yx}{(aa - xx)\sqrt{(aa - xx - yy)}} \text{ et}$$

$$\int x dx \left(\frac{d\Phi}{dx}\right) = y \int \frac{xxdx}{(aa - xx)\sqrt{(aa - xx - yy)}},$$

quo integrali inuento erit

$$z = ax \text{ Ang. fin. } \frac{y}{\sqrt{(aa - xx)}} - ay \int \frac{xxdx}{(aa - xx)\sqrt{(aa - xx - yy)}} + f:x + F:y$$

quae

quae forma per integrationem euoluta reducitur ad hanc

$$z = ax \,\text{Ang. sin.}\, \frac{y}{\sqrt{(aa-xx)}} + ay \,\text{Ang. sin.}\, \frac{x}{\sqrt{(aa-yy)}}$$
$$- aa \,\text{Ang. sin.}\, \frac{xy}{\sqrt{(aa-xx)(aa-yy)}} + f:x + F:y.$$

Formulae enim $\int \frac{aadx}{(aa-xx)\sqrt{(aa-xx-yy)}}$ integrale ita facillime elicitur. Ponatur $\frac{x}{\sqrt{(aa-xx-yy)}} = p$, erit $xx = \frac{pp(aa-yy)}{1+pp}$, et ob y constans per logarithmos differentiando

$$\frac{dx}{x} = \frac{dp}{p} - \frac{pdp}{1+pp} = \frac{dp}{p(1+pp)},$$

tum per illam formulam multiplicando

$$\frac{dx}{\sqrt{(aa-xx-yy)}} = \frac{dp}{1+pp}.$$

Porro est

$$aa - xx = \frac{aa+ppyy}{1+pp},$$

vnde formula integralis fit

$$\int \frac{aadx}{(aa-xx)\sqrt{(aa-xx-yy)}} = \int \frac{aadp}{aa+ppyy} = \frac{aa}{yy}\int \frac{dp}{\frac{aa}{yy}+pp}$$
$$= \frac{a}{y}\,\text{Ang. tang.}\, \frac{py}{a} = \frac{a}{y}\,\text{Ang. tang.}\, \frac{xy}{a\sqrt{(aa-xx-yy)}}$$
$$= \frac{a}{y}\,\text{Ang. sin.}\, \frac{xy}{\sqrt{(aa-xx)(aa-yy)}}.$$

Problema

Problema 45.

275. Si z eiusmodi esse debeat functio binarum variabilium x et y, vt sit

$$\left(\frac{ddz}{dx\,dy}\right) = P\left(\frac{dz}{dx}\right) + Q,$$

existentibus P et Q functionibus quibuscunque ipsarum x et y, inuestigare indolem functionis z.

Solutio.

Ponatur $\left(\frac{dz}{dx}\right) = v$, vt oriatur ista aequatio

$$\left(\frac{dv}{dy}\right) = Pv + Q$$

quae continet quantitates x, y et v, statuatur ergo x constans, eritque

$$dv = Pv\,dy + Q\,dy,$$

quae per $e^{-\int P dy}$ multiplicata praebet:

$$e^{-\int P dy}v = \int e^{-\int P dy}Q\,dy + f' : x,$$

ideoque

$$v = \left(\frac{dz}{dx}\right) = e^{\int P dy}\int e^{-\int P dy}Q\,dy + e^{\int P dy}f' : x.$$

Nunc cum haec integralia determinate contineant x et y, spectetur y vt constans, et sequens integratio praebet

$$z = \int e^{-\int P dy}dx\int e^{-\int P dy}Q\,dy + \int e^{\int P dy}dx f' : x + F : y$$

quae integralia quouis casu euoluta fiunt manifesta.

Coroll. 1.

276. Ad hoc ergo problema resoluendum per integrationem primo quaeratur R vt sit $\int P\,dy = lR$; deinde quaeratur S vt sit $\int \frac{Q\,dy}{R} = S$. Denique sit $\int RS\,dx = T$; ita vt in illis sola quantitas y hic vero sola x pro variabili habeatur. Quo facto erit nostrum integrale completum

$$z = T + \int R\,dx\,f':x + F:y.$$

Coroll. 2.

277. Hic ergo functio arbitraria $f:x$ in formula integrali est inuoluta, quae tamen si per applicatam curuae cuiuscumque respondentem abscissae x exhibeatur, hoc integrale $\int R\,dx\,f':x$ pro quouis valore ipsius y seorsim construi poterit, siquidem in hac integratione quantitas y vt constans spectatur.

Scholion.

278. Eodem plane modo resoluitur permutandis variabilibus x et y, hoc problema, quo functio z quaeritur, vt sit

$$\left(\frac{ddz}{dx\,dy}\right) = P\left(\frac{dz}{dy}\right) + Q,$$

dummodo P et Q sint functiones ipsarum x et y tantum, ipsam functionem z non implicantes; solutio enim ita se habebit

$$z = \int e^{\int P\,dx}\,dy \int e^{-\int P\,dx} Q\,dx + \int e^{\int P\,dx}\,dy\,f:y + F:x.$$

Quin

Quin etiam vtrumque problema latius extendi poteſt, ac prius reſolutionem admittet, ſi formula $\left(\frac{ddz}{dxdy}\right)$ aequetur functioni cuicunque trium quantitatum x, y et $\left(\frac{dz}{dx}\right)$, poſterius vero ſi $\left(\frac{ddz}{dxdy}\right)$ aequetur functioni cuicunque harum trium quantitatum x, y et $\left(\frac{dz}{dy}\right)$; vtroque enim caſu res reducitur ad aequationem differentialem primi gradus. Neque vero haec ſoluendi methodus ſuccedit, ſi vtraque formula primi gradus $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ ſimul ingrediatur, vel ſi functiones P et Q etiam ipſam quantitatem z complectantur.

Exemplum 1.

279. *Quaeratur binarum variabilium* x *et* y *functio* z, *vt ſit* $\left(\frac{ddz}{dxdy}\right) = \frac{n}{y}\left(\frac{dz}{dx}\right) + \frac{m}{x}$.

Sit $\left(\frac{dz}{dx}\right) = v$ erit

$$\left(\frac{dv}{dy}\right) = \frac{nv}{y} + \frac{m}{x},$$

et ſpectata x vt conſtante erit

$$dv = \frac{nvdy}{y} + \frac{mdy}{x},$$

vnde per y^n diuidendo prodit

$$\frac{v}{y^n} = \frac{m}{x}\int\frac{dy}{y^n} = \frac{-m}{(n-1)xy^{n-1}} + f':x$$

ita vt ſit

$$v = \left(\frac{dz}{dx}\right) = \frac{-my}{(n-1)x} + y^n f':x$$

 ſuma-

sumatur iam y constans, et denuo integrando obtinetur

$$z = \frac{-m}{n-1} y\, l\, x + y^n f : x + F : y.$$

Exemplum 2.

280. *Quaeratur binarum variabilium* x *et* y *functio* z, *vt sit* $\left(\frac{ddz}{dxdy}\right) = \frac{y}{xx+yy}\left(\frac{dz}{dx}\right) + \frac{a}{xx+yy}$.

Posito $\left(\frac{dz}{dx}\right) = v$ et sumto x constante erit

$$dv = \frac{vydy}{xx+yy} + \frac{ady}{xx+yy}$$

quae aequatio per $\sqrt{(xx+yy)}$ diuisa dat:

$$\frac{v}{\sqrt{(xx+yy)}} = a\int \frac{dy}{(xx+yy)^{\frac{3}{2}}} = \frac{ay}{xx\sqrt{(xx+yy)}} + f:x.$$

Ergo $v = \left(\frac{dz}{dx}\right) = \frac{ay}{xx} + \sqrt{(xx+yy)}.f:x$

sit iam y constans reperieturque

$$z = \frac{-ay}{x} + \int f:x.\,dx\sqrt{(xx+yy)} + F:y$$

vbi quidem integrale

$$\int f:x.\,dx\sqrt{(xx+yy)}$$

ob functionem indeterminatam $f:x$, etsi y constans ponitur, in genere exprimi nequit, ita vt explicate per y et functiones ipsius x exhiberi possit.

Scholion.

281. Formula ergo secundi gradus $\left(\frac{ddz}{dxdy}\right)$ non tam largam casuum resolubilium copiam admittit, quam

quam binae reliquae $\left(\frac{ddz}{dx^2}\right)$ et $\left(\frac{ddz}{dy^2}\right)$, cum in his ſolutio ſuccedat, etiamſi ipſa quantitas z quoque in earum determinationem ingrediatur, quod hic ſecus euenit, cum methodus non pateat huiusmodi aequationem $\left(\frac{ddz}{dxdy}\right)=P\left(\frac{dz}{dx}\right)+Q$, quando litterae P et Q quantitatem z continent reſoluendi; neque etiam ſolutio locum habet; quando praeter formulam primi gradus $\left(\frac{dz}{dx}\right)$ ſimul quoque altera adeſt. Interim tamen dantur caſus, quibus ſolutiones particulares exhiberi poſſunt, eaeque adeo infinitae, quae iunctim ſumtae ſolutioni generali aequiualere videntur, etiamſi in applicatione ad vſum practicum parum ſubſidii plerumque afferant; formas tamen huiusmodi ſolutionum notaſſe iuuabit.

Problema 46.

282. Si z eiusmodi debeat eſſe functio binarum variabilium x et y, vt fiat $\left(\frac{ddz}{dxdy}\right)=az$, indolem huius functionis z particulariter ſaltem inueſtigare.

Solutio.

Cum quantitas z vnam vbique teneat dimenſionem euidens eſt, ſi ſtatuatur $z=e^p q$, quantitatem exponentialem e^p ex calculo euaneſcere. Ponamus igitur $z=e^{\alpha x}Y$ ita vt Y functionem ipſius y tantum contineat, eritque

$$\left(\frac{dz}{dx}\right)=\alpha e^{\alpha x}Y \text{ et } \left(\frac{ddz}{dxdy}\right)=\alpha e^{\alpha x}\frac{dY}{dy}=a e^{\alpha x}Y,$$

 vnde

vnde fit

$$\frac{\alpha\, dY}{Y} = a\,dy \text{ et } Y = e^{\frac{ay}{\alpha}},$$

ſicque iam ſolutionem particularem habemus

$$z = A\, e^{\alpha x + \frac{ay}{\alpha}};$$

quae autem ſatis late patet, cum tam A quam α pro lubitu aſſumi poſſit. Plures autem valores ipſius z ſeorſim ſatisfacientes, etiam iunctim ſumti ſatisfaciunt, vnde huiusmodi expreſſionem multo generaliorem deducimus:

$$z = A e^{\alpha x + \frac{a}{\alpha} y} + B e^{\beta x + \frac{a}{\beta} y} + C e^{\gamma x + \frac{a}{\gamma} y} + D e^{\delta x + \frac{a}{\delta} y} \text{ etc.}$$

vbi cum A, B, C, etc. item α, β, γ, etc. omnes valores poſſibiles recipere queant, haec forma pro maxime vniuerſali eſt habenda, neque ſi ad amplitudinem ſpectamus, quicquam cedere videtur ſuperioribus ſolutionibus, quae binas functiones arbitrarias inuoluunt, propterea quod hic duplicis generis coefficientes arbitrarii occurrunt, interim tamen haud liquet, quomodo functiones diſcontinuae hac relatione repraeſentari queant.

Coroll. 1.

283. Pro ſolutione ergo particulari inuenienda, ſumantur bini numeri m et n, vt eorum productum ſit

fit $mn = a$, eritque $z = Ae^{mx+ny}$. Atque etiam ex iisdem numeris permutatis erit $z = Ae^{nx+my}$.

Coroll. 2.

284. Ex tali numerorum m et n pari vt fit $mn = a$, folutiones quoque per finus et cofinus exhiberi poffunt; erit enim

$$z = B\,\text{fin.}(mx - ny), \text{ vel } z = B\,\text{cof.}(mx - ny),$$

vel etiam permutando

$$z = B\,\text{fin.}(nx - my) \text{ vel } z = B\,\text{cof.}(nx - my).$$

Coroll. 3.

285. Cum igitur huiusmodi formulae innumerabiles exhiberi queant, fingulae per conftantes quascunque multiplicatae et in vnam fummam collectae dabunt folutionem generalem problematis.

Scholion.

286. Neque tamen haec folutio, etfi infinities infinitas determinationes recipit, ita eft comparata, vt eiusmodi folutionibus, quae binas functiones arbitrarias inuoluunt, aequiualens aeftimari poffit; propterea quod non patet, quomodo fingulas litteras affumi oporteat, vt pro dato cafu verbi gratia $y = 0$, quantitas z vel $(\frac{dz}{dx})$ feu $(\frac{dz}{dy})$ datae functioni ipfius x aequalis euadat, cuiuscunque etiam indolis fuerit

fuerit haec functio. Semper autem solutio generalis duplicis huiusmodi determinationis capax esse debet. Quando autem talem solutionem impetrare non licet, vtique eiusmodi solutionibus, vti hic inuenimus, contenti esse debemus. Ac tales quidem solutiones simili modo obtinere possumus si proponatur eiusmodi aequatio:

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+Rz=0$$

si modo litterae P, Q, R denotent functiones ipsius x tantum. Posito enim $z=e^{\alpha y}X$, vt X sit functio solius x, ob

$$\left(\frac{dz}{dx}\right)=e^{\alpha y}\frac{dX}{dx} \text{ et } \left(\frac{dz}{dy}\right)=\alpha e^{\alpha y}X \text{ erit}$$

$$\frac{\alpha dX}{dx}+\frac{PdX}{dx}+\alpha QX+RX=0,$$

vnde reperitur

$$\frac{dX}{X}=\frac{-dx(\alpha Q+R)}{\alpha+P}$$

sicque elicitur pro quouis numero α idoneus valor ipsius X. Quare sumendis infinitis numeris α, hoc modo expressio infinities infinitas determinationes recipiens colligitur:

$$z=Ae^{\alpha y}X+Be^{\beta y}X'+Ce^{\gamma y}X'' \text{ etc.}$$

Verum tamen dantur etiam casus eiusmodi aequationum, quae solutiones vere completas admittunt, quarum rationem in sequente problemate indagemus.

Problema

Problema 47.

287. Propofita aequatione refoluenda:

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+Rz+S=0$$

inueftigare cuiusmodi functiones ipfarum x et y effe debeant quantitates P, Q, R et S, vt haec aequatio folutionem vere completam admittat.

Solutio.

Sit V functio quaecunque ipfarum x et y, ac ponatur $z=e^V v$, ita vt iam v fit quantitas incognita, cuius valorem inueftigari oporteat. Cum igitur fit

$$\left(\frac{dz}{dx}\right)=e^V\left(\left(\frac{dv}{dx}\right)+v\left(\frac{dV}{dx}\right)\right);\ \left(\frac{dz}{dy}\right)=e^V\left(\left(\frac{dv}{dy}\right)+v\left(\frac{dV}{dy}\right)\right)$$

facta fubftitutione totaque aequatione per e^V diuifa prodibit fequens aequatio:

$$\left.\begin{aligned} e^{-V}S+\left(\frac{ddv}{dxdy}\right)+\left(\frac{dV}{dy}\right)\left(\frac{dv}{dx}\right)+\left(\frac{dV}{dx}\right)\left(\frac{dv}{dy}\right)+\left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right)v \\ +P\left(\frac{dv}{dx}\right)+Q\left(\frac{dv}{dy}\right)+\left(\frac{ddV}{dxdy}\right)v \\ +P\left(\frac{dV}{dx}\right)v \\ +Q\left(\frac{dV}{dy}\right)v \\ +\quad R\quad v \end{aligned}\right\}=0$$

Efficiendum iam eft, vt haec aequatio refolutionem completam admittat; cum igitur ante viderimus talem aequationem

$$\left(\frac{ddv}{dxdy}\right)+T\left(\frac{dv}{dx}\right)+e^{-V}S=0$$

 gene-

generaliter resolui posse, qualescunque etiam functiones ipsarum x et y pro S et V accipiantur, ad hanc aequationem illam redigamus. Necesse igitur est statui:

$$P+\left(\frac{dV}{dy}\right)=T\,;\ Q+\left(\frac{dV}{dx}\right)=0 \text{ et}$$

$$R+Q\left(\frac{dV}{dy}\right)+P\left(\frac{dV}{dx}\right)+\left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right)+\left(\frac{ddV}{dx\,dy}\right)=0$$

vnde obtinemus:

$$P=T-\left(\frac{dV}{dy}\right),\ Q=-\left(\frac{dV}{dx}\right) \text{ et}$$

$$R=\left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right)-T\left(\frac{dV}{dx}\right)-\left(\frac{ddV}{dx\,dy}\right).$$

Cum igitur per §. 275. reperiatur:

$$v=-\int e^{-\int T\,dy}dy\int e^{\int T\,dy-V}S\,dy+\int e^{-\int T\,dy}dx\,f:x+F:y$$

erit aequationis propositae

$$\left(\frac{ddz}{dx\,dy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+Rz+S=0$$

si modo litterae P, Q, R assignatos teneant valores, integrale completum:

$$z=-e^{V}\int e^{-\int T\,dy}dx\int e^{\int T\,dy-V}S\,dy+e^{V}\int e^{-\int T\,dy}dx\,f:x+e^{V}F:y$$

quandoquidem hic formae $f:x$ et $F:y$ functiones quascunque ipsius x et ipsius y denotant.

Coroll. 1.

288. Quaecunque ergo functiones ipsarum x et y pro litteris T et V accipiantur, inde oriuntur valores idonei pro litteris P, Q, R assumendi, vt aequa-

aequatio resolutionem completam admittat, functio autem S arbitrio nostro relinquitur.

Coroll. 2.

289. Possunt etiam in aequatione proposita functiones P et Q indefinitae relinqui, eritque tum

$$V = -\int Q dx \text{ et } \left(\frac{dV}{dy}\right) = -\int dx \left(\frac{dQ}{dy}\right), \text{ atque}$$
$$\left(\frac{ddV}{dx\,dy}\right) = -\left(\frac{dQ}{dy}\right);$$

vnde tantum quantitas R ita determinari debet, vt sit

$$R - PQ - \left(\frac{dQ}{dy}\right) = 0 \text{ seu}$$
$$R = PQ + \left(\frac{dQ}{dy}\right).$$

Coroll. 3.

290. Quia hic pro $\int Q dx$ scribi potest $\int Q dx + Y$, denotante Y functionem quamcunque ipsius y, ob $V = -\int Q dx - Y$ complete integrabilis erit haec aequatio:

$$\left(\frac{ddz}{dx\,dy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + \left(PQ + \left(\frac{dQ}{dy}\right)\right)z + S = 0$$

cuius integrale est

$$z = e^{-\int Q dx - Y} v,$$

existente

$$\left(\frac{ddv}{dx\,dy}\right) + \left(P - \int dx \left(\frac{dQ}{dy}\right) - \frac{dY}{dy}\right)\left(\frac{dv}{dx}\right) + e^{-Y} S = 0$$

existente

$$T = P - \int dx \left(\frac{dQ}{dy}\right) - \frac{dY}{dy},$$

ac propterea

$$\int T\,dy = \int P\,dy - \int Q\,dx - Y\,,$$

vnde valor ipsius v facile definitur.

Scholion.

291. In hoc calculo quo differentialia formularum integralium capi oportet, dum alia quantitas variabilis assumitur, atque in integratione supponitur, haec regula est tenenda, quod si fuerit $V = \int Q\,dx$, fore $\left(\frac{dV}{dy}\right) = \int dx \left(\frac{dQ}{dy}\right)$. Cum enim sit $\left(\frac{dV}{dx}\right) = Q$ erit $\left(\frac{ddV}{dx\,dy}\right) = \left(\frac{dQ}{dy}\right)$. Quodsi ergo statuatur $\left(\frac{dV}{dy}\right) = S$ erit $\left(\frac{dS}{dx}\right) = \left(\frac{dQ}{dy}\right)$, et $S = \left(\frac{dV}{dy}\right) = \int dx \left(\frac{dQ}{dy}\right)$; vnde vicissim colligitur si fuerit $S = \int dx \left(\frac{dQ}{dy}\right)$ fore ob $\int S\,dy = V$, integrando $\int S\,dy = \int Q\,dx$; quod cum ex principiis ante stabilitis per se sit manifestum, non opus esse iudico, pro hoc quasi nouo algorithmi genere praecepta seorsim tradere. Videamus autem in aliquot exemplis, cuiusmodi aequationes ope huius methodi complete resoluere liceat.

Exemplum 1.

292. *Proposita aequatione differentio-differentiali*

$$\left(\frac{ddz}{dx\,dy}\right) + a\left(\frac{dz}{dx}\right) + b\left(\frac{dz}{dy}\right) + Rz + S = 0$$

definire indolem functionis R, *vt haec aequatio resolutionem admittat, existente* S *functione quacunque ipsarum* x *et* y.

Cum

Cum sit $P=a$, et $Q=b$ erit $R=ab$, et $V=-bx$ tuto enim functio Y omitti potest, quia in sequente integratione iam binae functiones arbitrariae introducuntur, erit $T=a$. Vnde posito $z=e^{-bx}v$, habebitur haec aequatio

$$\left(\frac{ddv}{dxdy}\right)+a\left(\frac{dv}{dx}\right)+e^{bx}S=0$$

ac posito $\left(\frac{dv}{dx}\right)=u$ fit

$$\left(\frac{du}{dy}\right)+au+e^{bx}S=0,$$

et sumto x constante

$$e^{ay}u=-\int e^{ay+bx}S\,dy+f':x \text{ ergo}$$

$$u=\left(\frac{dv}{dx}\right)=-e^{-ay}\int e^{ay+bx}S\,dy+e^{-ay}f':x$$

et sumto iam y constante:

$$v=-e^{-ay}\int dx\int e^{ay+bx}S\,dy+e^{-ay}f:x+F:y$$

sumendo

$$\int dx\,f':x=f:x.$$

Quod si iam pro $e^{-bx}f:x$ scribatur $f:x$ erit

$$z=-e^{-ay-bx}\int dx\int e^{ay+bx}S\,dy+e^{-ay}f:x+e^{-bx}F:y.$$

Aliter.

Si sumsissemus $V=-bx-ay$, prodiisset $T=a-a=0$; ideoque posito $z=e^{-bx-ay}v$, quantitas v ex hac aequatione

$$\left(\frac{ddv}{dxdy}\right)+e^{bx+ay}S=0,$$

 defi-

definiri deberet quae dat

$$\left(\frac{dv}{dx}\right) = -\int e^{bx+ay}\, S\, dy + f' : x \text{ et}$$

$$v = -\int dx \int e^{bx+ay}\, S\, dy + f : x + F : y \text{ et}$$

$$z = e^{-bx-ay}\left(-\int dx \int e^{bx+ay} S\, dy + f : x + F : y\right),$$

quae forma simplicior est praecedente, etiamsi eodem redeat, estque hoc integrale completum aequationis:

$$\left(\frac{ddz}{dxdy}\right) + a\left(\frac{dz}{dx}\right) + b\left(\frac{dz}{dy}\right) + abz + S = 0.$$

Exemplum 2.

293. *Proposita aequatione differentio-differentiali:*

$$\left(\frac{ddz}{dxdy}\right) + \frac{a}{y}\left(\frac{dz}{dx}\right) + \frac{b}{x}\left(\frac{dz}{dy}\right) + Rz + S = 0$$

definire indolem functionis R, *vt haec aequatio resolutionem admittat, existente* S *functione quacunque ipsarum* x *et* y.

Cum sit $P = \frac{a}{y}$ et $Q = \frac{b}{x}$, erit $V = -blx - Y$, hincque $R = \frac{ab}{xy}$, et aequatio integrabilis erit

$$\left(\frac{ddz}{dxdy}\right) + \frac{a}{y}\left(\frac{dz}{dx}\right) + \frac{b}{x}\left(\frac{dz}{dy}\right) + \frac{ab}{xy}z + S = 0.$$

Quoniam igitur fit

$$T = P + \left(\frac{dV}{dy}\right) = \frac{a}{y} - \frac{dY}{dy},$$

sumamus $Y = +aly$, vt fiat $T = 0$, ac posito

$$z = e^{-blx-aly}v = x^{-b}y^{-a}v,$$

quan-

quantitas v ex hac aequatione definiri debet:

$$\left(\frac{ddv}{dxdy}\right) + x^b y^a S = 0,$$

vnde fit

$$\left(\frac{dv}{dx}\right) = -x^b \int y^a S\, dy + f' : x \text{ et}$$

$$v = -\int x^b dx \int y^a S\, dy + f : x + F : y$$

ideoque

$$z = \frac{-\int x^b dx \int y^a S dy + f : x + F : y}{x^b y^a}.$$

Scholion 1.

294. Hinc igitur patet hanc aequationem ope istius methodi in genere integrari posse:

$$\left(\frac{ddz}{dxdy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + \left(PQ + \left(\frac{dQ}{dy}\right)\right) z + S = 0$$

quaecunque functiones ipsarum x et y pro P, Q et S accipiantur. Ac resolutio quidem ita se habet, vt posito $z = e^{-\int Q dx - Y} v$, haec quantitas v determinetur hac aequatione:

$$\left(\frac{ddv}{dxdy}\right) + \left(P - \int dx \left(\frac{dQ}{dy}\right) - \frac{dY}{dy}\right)\left(\frac{dv}{dx}\right) + e^{\int Q dx + Y} S = 0$$

vbi iam pro Y talis functio ipsius y accipi potest, vt huius aequationis forma simplicissima euadat; id quod potissimum euenit si expressio

$$P - \int dx \left(\frac{dQ}{dy}\right) - \frac{dY}{dy}$$

ad

ad nihilum redigi queat. In genere autem reperitur

$$v = -\int e^{-\int P dy + \int Q dx + Y} dx \int e^{\int P dy} S dy + \int e^{-\int P dy + \int Q dx + Y} dx f : x + F : Y$$

qui valor ergo per $e^{-\int Q dx - Y}$ multiplicatus praebet formam functionis z. Hoc modo autem functio Y ab arbitrio nostro pendens penitus e calculo egreditur, fitque

$$z = -e^{-\int Q dx} \int e^{-\int P dy + \int Q dx} dx \int e^{\int P dy} S dy$$
$$+ e^{-\int Q dx} \int e^{-\int P dy + \int Q dx} dx f : x + e^{-\int Q dx} F : y$$

quod est integrale completum huius aequationis

$$\left(\frac{ddz}{dxdy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + \left(PQ + \left(\frac{dQ}{dy}\right)z + S = 0.$$

Scholion 2.

295. Permutandis autem variabilibus x et y etiam haec aequatio complete integrari potest:

$$\left(\frac{ddz}{dxdy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + \left(PQ + \left(\frac{dP}{dx}\right)\right)z + S = 0$$

cuius integrale erit:

$$z = -e^{-\int P dy} \int e^{-\int Q dx + \int P dy} dy \int e^{\int Q dx} S dx$$
$$+ e^{-\int P dy} \int e^{-\int Q dx + \int P dy} dy f : y + e^{-\int P dy} F : x$$

vbi praecipue hic casus in vtraque forma contentus notari meretur, si fuerit $P = Y$ et $Q = X$, existente X functione ipsius x et Y ipsius y tantum; tum enim huius aequationis

$$\left(\frac{ddz}{dxd}\right) + Y\left(\frac{dz}{dx}\right) + X\left(\frac{dz}{dy}\right) + XYz + S = 0$$

inte-

integrale completum erit

$$z = -e^{-\int X dx - \int Y dy} \int e^{\int X dx} dx \int e^{\int Y dy} S dy + e^{-\int X dx - \int Y dy} (f:x + F:Y)$$

quod etiam ita exhiberi poteſt:

$$e^{\int X dx + \int Y dy} z = f:x + F:y - \int e^{\int X dx} dx \int e^{\int Y dy} S dy$$

vel etiam hoc modo

$$e^{\int X dx + \int Y dy} z = f:x + F:y - \int e^{\int Y dy} dy \int e^{\int X dx} S dx$$

CAPVT III.

SI DVAE VEL OMNES FORMVLAE SECVNDI GRADVS PER RELIQVAS QVANTITATES DETERMINANTVR.

Problema 48.

246.

Si z eiusmodi debeat esse functio ipsarum x et y, vt fiat $\left(\frac{ddz}{dy^2}\right) = aa\left(\frac{ddz}{dx^2}\right)$, indolem functionis z determinare.

Solutio.

Introducantur binae nouae variabiles t et u, vt sit $t = \alpha x + \beta y$ et $u = \gamma x + \delta y$, atque ex §. 231. omnes formulae differentiales sequentes mutationes subibunt:

$$\left(\frac{dz}{dx}\right) = \alpha\left(\frac{dz}{dt}\right) + \gamma\left(\frac{dz}{du}\right);\ \left(\frac{dz}{dy}\right) = \beta\left(\frac{dz}{dt}\right) + \delta\left(\frac{dz}{du}\right)$$

$$\left(\frac{ddz}{dx^2}\right) = \alpha\alpha\left(\frac{ddz}{dt^2}\right) + 2\alpha\gamma\left(\frac{ddz}{dtdu}\right) + \gamma\gamma\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dxdy}\right) = \alpha\beta\left(\frac{ddz}{dt^2}\right) + (\alpha\delta + \beta\gamma)\left(\frac{ddz}{dtdu}\right) + \gamma\delta\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dy^2}\right) = \beta\beta\left(\frac{ddz}{dt^2}\right) + 2\beta\delta\left(\frac{ddz}{dtdu}\right) + \delta\delta\left(\frac{ddz}{du^2}\right)$$

vnde

vnde noſtra aequatio tranſibit in hanc:

$$(\beta\beta-\alpha\alpha aa)\left(\frac{ddz}{dt^2}\right)+2(\beta\delta-\alpha\gamma aa)\left(\frac{ddz}{dt\,du}\right)$$
$$+(\delta\delta-\gamma\gamma aa)\left(\frac{ddz}{du^2}\right)=0$$

ponatur ergo

$\beta\beta=\alpha\alpha aa$ et $\delta\delta=\gamma\gamma aa$, ſeu

$\alpha=1$; $\gamma=1$; $\beta=a$ et $\delta=-a$,

vt binae formulae extremae euaneſcant quod fit ponendo

$t=x+ay$ et $u=x-ay$,

eritque

$-2(aa+aa)\left(\frac{ddz}{dt\,du}\right)=0$ ſeu $\left(\frac{ddz}{dt\,du}\right)=0$

vnde per §. 269. colligitur integrale completum

$z=f:t+F:u$

ac pro t et u reſtitutis valoribus:

$z=f:(x+ay)+F:(x-ay)$

quae forma manifeſto ſatisfacit cum ſit

$\left(\frac{dz}{dx}\right)=f':(x+ay)+F':(x-ay)$; $\left(\frac{dz}{dy}\right)=af':(x+ay)$
$-aF':(x-ay)$

$\left(\frac{ddz}{dx^2}\right)=f'':(x+ay)+F'':(x-ay)$; $\left(\frac{ddz}{dy^2}\right)=aaf'':(x+ay)$
$+aaF'':(x-ay)$.

 Coroll. 1.

Coroll. 1.

297. Valor igitur ipſius z aequatur aggregato duarum functionum arbitrariarum alterius ipſius $x+ay$, alterius ipſius $x-ay$, atque ambae hae functiones ita ad arbitrium aſſumi poſſunt, vt etiam functiones diſcontinuas earum loco capere liceat.

Coroll. 2.

298. Pro lubitu ergo binae curuae quaecunque etiam libero manus tractu deſcriptae ad hunc vſum adhiberi poſſunt. Scilicet ſi in vna abſciſſa capiatur $=x+ay$, in altera vero abſciſſa $=x-ay$, ſumma applicatarum ſemper valorem idoneum pro functione z ſuppeditabit.

Scholion 1.

299. Hoc fere primum eſt problema, quod in hoc nouo calculi genere ſoluendum occurrit; perduxerat autem ſolutio generalis problematis de cordis vibrantibus ad hanc ipſam aequationem, quam hic tractauimus. Celeb. *Alembertus*, qui hoc problema primus felici ſucceſſu eſt aggreſſus, methodo ſingulari aequationem integrauit; ſcilicet cum eſſe oporteat $\left(\frac{ddz}{dy^2}\right)=a^2\left(\frac{ddz}{dx^2}\right)$, poſito $dz=pdx+qdy$, indeque $dp=rdx+sdy$ et $dq=sdx+tdy$, illa aequatio

quatio poſtulat vt ſit $t = aar$. Conſideratis porro iſtis aequationibus

$$\left.\begin{array}{l} dp = rdx + sdy \\ dq = sdx + aardy \end{array}\right| \begin{array}{l} \text{elicitur combinando} \\ adp + dq = ar(dx + ady) + s(ady + dx) \end{array}$$

ſeu $adp + dq = (ar + s)(dx + ady)$

vnde patet $ar + s$ functioni ipſius $x + ay$ aequari debere ex quo etiam $ap + q$ tali functioni aequatur. Atque quia a aeque negatiue ac poſitiue accipi poteſt, habentur duae huiusmodi aequationes:

$$ap + q = 2af' : (x + ay) \text{ et } q - ap = 2aF' : (x - ay)$$

vnde colligitur:

$$q = af' : (x + ay) + aF' : (x - ay) \text{ et}$$

$$p = f' : (x + ay) - F' : (x - ay)$$

hincque aequatio $dz = pdx + qdy$ ſponte integratur, fitque

$$z = f : (x + ay) - F : (x - ay).$$

Hoc modo ſagaciſſimus Vir integrale completum eſt adeptus, ſed non animaduertit, loco functionum harum introductarum, non ſolum omnis generis functiones continuas ſed etiam omni continuitatis lege deſtitutas accipi licere.

Scholion 2.

300. Cum plurimum interſit, in hoc nouo calculi genere quam plurimas methodos perſequi, ab aliis ſolutio noſtrae aequationis ita eſt tentata,

vt ponerent $\left(\frac{dz}{dy}\right)=k\left(\frac{dz}{dx}\right)$, vnde fit primo $\left(\frac{ddz}{dx\,dy}\right)=k\left(\frac{ddz}{dx^2}\right)$, tum vero $\left(\frac{ddz}{dy^2}\right)=k\left(\frac{ddz}{dx\,dy}\right)$ ex quo colligitur $\left(\frac{ddz}{dy^2}\right)=kk\left(\frac{ddz}{dx^2}\right)$. Euidens ergo eſt pro noſtro caſu capi debere $kk=aa$ ſeu $k=\pm a$. Sit ergo $k=a$, et ob $\left(\frac{dz}{dy}\right)=a\left(\frac{dz}{dx}\right)$, fiet

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right)=\left(\frac{dz}{dx}\right)(dx+ady)\quad.$$

hincque manifeſtum eſt fore $z=f:(x+ay)$, et ob a ambiguum, quoniam bini valores ſeorſim ſatisfacientes etiam iuncti ſatisfacient, concluditur ipſa ſolutio inuenta. Hoc etiam modo negotium confici poteſt. Statuatur

$$\left(\frac{ddz}{dy^2}\right)=aa\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddv}{dx\,dy}\right)$$

eritque

$$\left(\frac{dz}{dy}\right)=\left(\frac{dv}{dx}\right)\text{ et }aa\left(\frac{dz}{dx}\right)=\left(\frac{dv}{dy}\right).$$

Inuentis nunc formulis primi gradus $\left(\frac{dv}{dx}\right)$ et $\left(\frac{dv}{dy}\right)$ ob

$$dv=dx\left(\frac{dv}{dx}\right)+dy\left(\frac{dv}{dy}\right),$$

habebimus has aequationes:

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right)\text{ et}$$

$$dv=dx\left(\frac{dz}{dy}\right)+aa\,dy\left(\frac{dz}{dx}\right)$$

ex quarum combinatione colligimus:

$$dv+adz=(dx+ady)\left(\left(\frac{dz}{dy}\right)+a\left(\frac{dz}{dx}\right)\right),$$

hincque

$$v+az=f:(x+ay)\text{ et }v-az=F:(x-ay)\quad.$$

ſicque

ſicque pro z eadem forma exſurgit. Methodus vero, quam in ſolutione ſum ſecutus, ad naturam rei magis videtur accommodata, cum etiam in aliis problematibus magis complicatis inſignem vtilitatem afferat.

Scholion 3.

301. Noſtra autem ſolutio hoc habet incommodi, quod pro hac aequatione $\left(\frac{ddz}{dy^2}\right)+aa\left(\frac{ddz}{dx^2}\right)=0$ ad expreſſionem imaginariam deducit ſcilicet:

$$z=f:(x+ay\sqrt{-1})+F:(x-ay\sqrt{-1})$$

quoties autem functiones f et F ſunt continuae, cuiuscunque demum fuerint indolis, ſemper earum valores ad hanc formam $P\pm Q\sqrt{-1}$ reduci poſſunt, vnde ſequens forma ex illa facile deducenda ſemper valorem realem exhibebit:

$$z=\tfrac{1}{2}f:(x+ay\sqrt{-1})+\tfrac{1}{2}f:(x-ay\sqrt{-1})+\frac{1}{2\sqrt{-1}}F:(x+ay\sqrt{-1})$$
$$-\frac{1}{2\sqrt{-1}}F:(x-ay\sqrt{-1})$$

pro cuius ad realitatem reductione notaſſe iuuabit, poſito

$x=s\,\text{coſ}.\Phi$ et $ay=s\,\text{ſin}.\Phi$ fore:

$$(x\pm ay\sqrt{-1})^n=s^n(\text{coſ}.n\Phi\pm\sqrt{-1}.\text{ſin}.n\Phi).$$

Quare quoties functiones propoſitae per operationes analyticas ſunt conflatae, hoc eſt, continuae, earum valores realiter per coſinus et ſinus angulorum multiplorum ipſius Φ exhiberi poſſunt. Quando autem

autem functiones illae sunt discontinuae, talis reductio neutiquam locum habet, etiamsi certum videatur, etiam tunc formam allatam valorem realem esse adepturam. Quis autem in curua quaecunque libero manus ductu descripta applicatas abscissis $x+ay\sqrt{-1}$ et $x-ay\sqrt{-1}$ respondentes animo saltem imaginari, ac summam earum realem assignare valuerit; aut differentiam, quae per $\sqrt{-1}$ diuisa etiam erit realis? Hic ergo haud exiguus defectus calculi cernitur, quem nullo adhuc modo supplere licet; atque ob hunc ipsum defectum huiusmodi solutiones vniuersales plurimum de sua vi perdunt.

Problema 49.

302. Proposita aequatione $\left(\frac{ddz}{dy^2}\right)=PP\left(\frac{ddz}{dx^2}\right)$, inquirere, quales functiones ipsarum x et y pro P assumere liceat, vt integratio ope reductionis succedat.

Solutio.

Reductionem hanc ita fieri assumo, vt loco x et y, binae aliae variabiles t et u introducantur, qua substitutione secundum §. 231. in genere facta prodit haec aequatio:

$$\left.\begin{array}{l} +\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right) \\ -PP\left(\frac{ddt}{dx^2}\right)-PP\left(\frac{ddu}{dx^2}\right)-PP\left(\frac{dt}{dx}\right)^2-2PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)-PP\left(\frac{du}{dx}\right)^2 \end{array}\right\}=0$$

Iam relatio inter binas variabiles t, u et praecedentes

dentes x, y eiusmodi statuatur vt binae formulae $\left(\frac{ddz}{dt^2}\right)$ et $\left(\frac{ddz}{du^2}\right)$ ex calculo egrediantur, id quod fiet ponendo

$$\left(\frac{dt}{dy}\right)+P\left(\frac{dt}{dx}\right)=0 \text{ et } \left(\frac{du}{dy}\right)-P\left(\frac{du}{dx}\right)=0.$$

Tum autem erit

$$\left(\frac{ddt}{dy^2}\right)=-P\left(\frac{ddt}{dxdy}\right)-\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right),$$

at cum sit indidem

$$\left(\frac{ddt}{dxdy}\right)=-P\left(\frac{ddt}{dx^2}\right)-\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right), \text{ erit}$$

$$\left(\frac{ddt}{dy^2}\right)=PP\left(\frac{ddt}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)-\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right)$$

similique modo sumendo P negatiue

$$\left(\frac{ddu}{dy^2}\right)=PP\left(\frac{ddu}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{du}{dx}\right)+\left(\frac{dP}{dy}\right)\left(\frac{du}{dx}\right).$$

Quibus substitutis nostra aequatio hanc induet formam:

$$\left[P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)\right]\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)+\left[P\left(\frac{dP}{dx}\right)+\left(\frac{dP}{dy}\right)\right]\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right)$$
$$-4PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right)=0$$

quae cum vnicam formulam secundi gradus $\left(\frac{ddz}{dtdu}\right)$ contineat integrationem admittit si vel $\left(\frac{dz}{dt}\right)$ vel $\left(\frac{dz}{du}\right)$ e calculo excesserit. Ponamus ergo insuper

$$P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)=0,$$

qua aequatione indoles quaesitae functionis P definitur; quo facto aequatio integranda per $2P\left(\frac{du}{dx}\right)$ diuisa erit

$$\left(\frac{dP}{dx}\right)\left(\frac{dz}{du}\right)-2P\left(\frac{dt}{dx}\right)\left(\frac{ddz}{dtdu}\right)=0$$

 cuius

cuius integrale posito $\left(\frac{dz}{du}\right)=v$, fit

$$2\,lv=\int\frac{dt\left(\frac{dP}{dx}\right)}{P\left(\frac{dt}{dx}\right)}=l\left(\frac{dz}{du}\right).$$

Verum prius ipsam functionem P per x et y definiri oportet. Cum sit $\left(\frac{dP}{dy}\right)=P\left(\frac{dP}{dx}\right)$ erit

$$dP=dx\left(\frac{dP}{dx}\right)+P\,dy\left(\frac{dP}{dx}\right),$$

hincque ponendo breuitatis ergo $\left(\frac{dP}{dx}\right)=p$, fit

$$dx=\frac{dP}{p}-P\,dy,\ \text{atque}$$

$$x=-Py+\int dP\left(y+\frac{1}{p}\right).$$

Statuatur ergo $y+\frac{1}{p}=f':P$, ac reperitur

$$x+Py=f:P\ \text{et}\ p=\left(\frac{dP}{dx}\right)=\frac{1}{f':P-y},$$

ac $\left(\frac{dP}{dy}\right)=\frac{P}{f':P-y}$ vnde ratio determinationis quantitatis P per x et y definitur. Pro nouis autem variabilibus t et u, ob $\left(\frac{dt}{dy}\right)=-P\left(\frac{dt}{dx}\right)$ erit

$$dt=\left(\frac{dt}{dx}\right)(dx-P\,dy)\ \text{et ob}\ x=-Py+f:P\ \text{fit}$$

$$dt=\left(\frac{dt}{dx}\right)(dPf':P-2P\,dy-y\,dP)=P^{\frac{1}{2}}\left(\frac{dt}{dx}\right)\left(\frac{dP}{\sqrt{P}}f':P-2dy\sqrt{P}-\frac{y\,dP}{\sqrt{P}}\right)$$

cuius postremae formulae cum integrale sit

$$\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\ \text{erit}$$

$$t=F:\left(\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\right).$$

Deinde ob $\left(\frac{du}{dy}\right)=P\left(\frac{du}{dx}\right)$ habetur

$$du=\left(\frac{du}{dx}\right)(dx+P\,dy)=\left(\frac{du}{dx}\right)(dPf':P-y\,dP),$$

ideoque

ideoque

$$du = \left(\frac{dz}{dx}\right)(f':P - y)\,dP;$$

quare u aequabitur functioni ipsius P. In hoc autem negotio functiones quascunque accipere licet, quia sequente demum integratione vniuersalitas solutionis obtinetur. Quare ponamus:

$$t = \int \frac{dP}{\sqrt{P}} f':P - 2y\sqrt{P} \quad \text{et} \quad u = P \ \text{existente}$$

$$x + Py = f:P.$$

Denique ad ipsum integrale inueniendum, quia est

$$2\,l\left(\frac{dz}{du}\right) = \int \frac{dt\left(\frac{dP}{dx}\right)}{P\left(\frac{dt}{dx}\right)}$$

in qua integratione u seu P sumitur constans, per superiora est

$$\frac{dt}{\left(\frac{dt}{dx}\right)} = dP f':P - 2P\,dy - y\,dP = -2P\,dy$$

ob P constans, et $\left(\frac{dP}{dx}\right) = \frac{1}{f':P - y}$, vnde fit

$$2\,l\left(\frac{dz}{dP}\right) = \int \frac{-2\,dy}{f':P - y} = 2\,l(f':P - y) + 2\,lF:P \quad \text{seu}$$

$$\left(\frac{dz}{dP}\right) = (f':P - y)\,F:P,$$

hincque porro

$$z = \int dP\,(f':P - y)\,F:P$$

sumendo hic t constans. Cum igitur sit

$$y = +\frac{1}{2\sqrt{P}}\int \frac{dP}{\sqrt{P}} f':P - \frac{t}{2\sqrt{P}}$$

ideoque

$$f':P - y = f':P - \frac{1}{2\sqrt{P}} \int \frac{dP}{\sqrt{P}} f':P + \frac{t}{2\sqrt{P}},$$

vnde conficitur

$$z = \int dP \left(f':P - \frac{1}{2\sqrt{P}} \int \frac{dP}{\sqrt{P}} f':P \right) F:P + \left(\tfrac{1}{2} \int \frac{dP}{\sqrt{P}} f':P - y\sqrt{P} \right) \int \frac{dP}{\sqrt{P}} F:P$$
$$+ \Phi : \left(\int \frac{dP}{\sqrt{P}} f':P - 2y\sqrt{P} \right)$$

quae expressio duas continet functiones arbitrarias F et Φ.

Coroll. 1.

303. Primum huius formae membrum ita transformari potest:

$$\int \frac{dP}{\sqrt{P}} \left(\sqrt{P}.f':P - \tfrac{1}{2} \int \frac{dP}{\sqrt{P}} f':P \right) F:P, \text{ at}$$

$$\sqrt{P}.f':P - \tfrac{1}{2} \int \frac{dP}{\sqrt{P}} f':P = \int dP \sqrt{P}.f'':P,$$

vnde primum membrum erit

$$\int \frac{dP}{\sqrt{P}} F:P. \int dP \sqrt{P}.f'':P.$$

Coroll. 2.

304. Cum autem hoc primum membrum sit functio indefinita ipsius P si ea indicetur per Π:P, erit

$$\frac{dP}{\sqrt{P}} F:P = \frac{dP\,\Pi':P}{\int dP \sqrt{P}.f'':P},$$

vnde forma integralis fit

$$z = \Pi:P + \Phi : \left(\int \frac{dP}{\sqrt{P}} f':P - 2y\sqrt{P} \right) + \left(\int \frac{dP}{\sqrt{P}} f':P - 2y\sqrt{P} \right)$$
$$\int \frac{dP\,\Pi':P}{2\int dP \sqrt{P}.f'':P}.$$

Coroll. 3.

Coroll. 3.

305. Solutio magis particularis nafcitur fumendo $\Pi : P = 0$ hincque z aequabitur functioni cuicunque quantitatis $\int \frac{dP}{\sqrt{P}} f' : P - 2y\sqrt{P}$, quae ob $x + Py = f : P$ per x et y exhiberi cenfenda eft.

Scholion.

306. Quanquam hic eadem methodo fum vfus atque in problemate praecedente, tamen quod mirum videatur, cafus praecedentis problematis quo erat $P = a$ in hac folutione non continetur. Ratio huius paradoxi in refolutione aequationis $\left(\frac{dP}{dy}\right) = P\left(\frac{dP}{dx}\right)$ eft fita cui manifefto fatisfacit valor $P = a$, etiamfi in forma inde deriuata $x + Py = f : P$ non contineatur. Hic fcilicet fimile quiddam vfu venit, quod iam fupra obferuauimus, faepe aequationi differentiali valorem quendam fatisfacere poffe, qui in integrali non contineatur, veluti aequationi $dy\sqrt{(a-x)} = dx$ fatisfacere videmus valorem $x = a$, quem tamen integralis $y = C - 2\sqrt{(a-x)}$ excludit. Quare etiam noftro cafu valor $P = a$ peculiarem euolutionem poftulat in priore problemate peractam de reliquis, vbi pro $f : P$ certa quaedam functio ipfius P affumitur exempla quaedam euoluamus.

Exemplum 1.

307. *Sumto* $f:P=0$, *vt fit* $P=-\frac{x}{y}$, *integrale completum huius aequationis*

$$\left(\frac{ddz}{dy^2}\right)=\frac{xx}{yy}\left(\frac{ddz}{dx^2}\right)$$

inuestigare.

Cum fit $f':P=0$, folutio inuenta ob $\int\frac{dP}{\sqrt{P}}f':P=C$ praebet

$$z=\frac{-C}{2}\int\frac{dP}{\sqrt{P}}F:P+(\tfrac{1}{2}C-y\sqrt{P})\int\frac{dP}{\sqrt{P}}F:P+\Phi:(C-2y\sqrt{P}).$$

Statuatur $\int\frac{dP}{\sqrt{P}}F:P=\Pi:P$ prodibitque:

$$z=-y\sqrt{P}.\Pi:P+\Phi:y\sqrt{P}.$$

Reftituatur pro P valor $\frac{-x}{y}$, et ob $y\sqrt{P}=\sqrt{-xy}$, imaginarium $\sqrt{-1}$ in functiones inuoluendo erit:

$$z=\sqrt{xy}.\Pi:\frac{x}{y}+\Phi:\sqrt{xy}$$

quae forma facile in hanc transfunditur:

$$z=x\Gamma:\frac{x}{y}+\Theta:xy$$

vbi $x\Gamma:\frac{x}{y}$ denotat functionem quamcunque homogeneam vnius dimenfionis ipfarum x et y. Refolutio autem inftituetur loco x et y has nouas variabiles t et u introducendo vt fit $t=C-2\sqrt{-xy}$ et $u=-\frac{x}{y}$ vel etiam fimplicius $t=2\sqrt{xy}$ et $u=\frac{x}{y}$, vnde fit:

$$\left(\frac{dt}{dx}\right)=\frac{\sqrt{y}}{\sqrt{x}};\ \left(\frac{dt}{dy}\right)=\frac{\sqrt{x}}{\sqrt{y}};\ \left(\frac{ddt}{dx^2}\right)=\frac{-\sqrt{y}}{2x\sqrt{x}};\ \left(\frac{ddt}{dy^2}\right)=\frac{-\sqrt{x}}{2y\sqrt{y}}.$$

$$\left(\frac{du}{dx}\right)=\frac{1}{y};\ \left(\frac{du}{dy}\right)=\frac{-x}{yy};\ \left(\frac{ddu}{dx^2}\right)=0;\ \left(\frac{ddu}{dy^2}\right)=\frac{2x}{y^3}.$$

et

et ob $PP = \frac{xx}{yy}$ aequatio proposita hanc induit formam:

$$0\left(\frac{dz}{dt}\right) + \frac{2x}{y^3}\left(\frac{dz}{du}\right) - \frac{4x\sqrt{x}}{yy\sqrt{y}}\left(\frac{ddz}{dt\,du}\right) = 0.$$

Nunc cum sit

$$ttu = 4xx\text{, et } x = \tfrac{1}{2}t\sqrt{u}\text{,}$$

atque $y = \frac{t}{2\sqrt{u}}$, habebimus:

$$\frac{8uu}{tt}\left(\frac{dz}{du}\right) - \frac{8uu}{t}\left(\frac{ddz}{dt\,du}\right) = 0 \text{ seu } \left(\frac{dz}{du}\right) = t\left(\frac{ddz}{dt\,du}\right).$$

Fiat $\left(\frac{dz}{du}\right) = v$, vt sit $v = t\left(\frac{dv}{dt}\right)$ et sumto u constante $\frac{dt}{t} = \frac{dv}{v}$, ergo $v = \left(\frac{dz}{du}\right) = tf':u$. Sit iam t constans fietque

$$z = tf:u + F:t = 2\sqrt{xy}.f:\frac{x}{y} + F:\sqrt{xy}$$

vt ante.

Corollarium.

308. Quemadmodum autem expressio inuenta $z = x\Gamma:\frac{x}{y} + \Theta:xy$ satisfaciat, differentialibus rite sumtis perspicietur:

$$\left(\frac{dz}{dx}\right) = \Gamma:\frac{x}{y} + \frac{x}{y}F':\frac{x}{y} + y\Theta':xy;\ \left(\frac{dz}{dy}\right) = \frac{-xx}{yy}\Gamma':\frac{x}{y} + x\Theta':xy$$

vnde porro fit:

$$\left(\frac{ddz}{dx^2}\right) = \frac{2}{y}\Gamma':\frac{x}{y} + \frac{x}{yy}\Gamma'':\frac{x}{y} + yy\Theta'':xy \text{ et}$$

$$\left(\frac{ddz}{dy^2}\right) = \frac{2xx}{y^3}\Gamma':\frac{x}{y} + \frac{x^3}{y^4}\Gamma'':\frac{x}{y} + xx\Theta'':xy.$$

Exemplum 2.

309. *Sumto* $f:P = \frac{PP}{2a}$, *et sit*

$$PP = 2aPy + 2ax \quad et \quad P = ay + \sqrt{(aayy + 2ax)},$$

huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = (2aayy + 2ax + 2ay\sqrt{(aayy + 2ax)})\left(\frac{ddz}{dx^2}\right)$$

integrale completum inuestigare.

Cum sit $f:P = \frac{PP}{2a}$ erit

$$f':P = \frac{P}{a}, \quad \text{et} \int \frac{dP}{\sqrt{P}} f':P = \int \frac{1}{a} dP\sqrt{P} = \frac{2}{3a} P\sqrt{P},$$

vnde forma generalis supra inuenta abit in

$$z = \int dP.\frac{2P}{3a}F:P + \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right)\int \frac{dP}{\sqrt{P}}F:P + \Phi:\left(\frac{2}{3a}P\sqrt{P} - 2y\sqrt{P}\right)$$

statuatur $\int \frac{dP}{\sqrt{P}}.F:P = \Pi:P$ erit

$$dP.F:P = dP\sqrt{P}.\Pi'P$$

atque

$$z = \frac{2}{3a}\int P^{\frac{3}{2}} dP.\Pi':P + \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right)\Pi:P + \Phi:\left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right).$$

Est autem

$$\frac{P}{3a} - y = \frac{-2}{3}y + \frac{1}{3}\sqrt{\left(yy + \frac{2x}{a}\right)};$$

quarum formularum euolutio deducit ad expressiones nimis perplexas. At substitutiones ad scopum perducentes sunt

$$t = \frac{2}{3a}P\sqrt{P} - 2y\sqrt{P} \quad \text{et} \quad u = P.$$

Corol-

Corollarium.

310. Si pro solutione magis restricta ponatur

$$\Pi : P = P^{n-\frac{1}{2}} \text{ erit}$$

$$\Pi' : P = (n-\tfrac{1}{2}) P^{n-\frac{3}{2}},$$

hincque colligitur

$$z = \frac{n}{(n+1)a} P^{n+1} - P^n y + \Phi : \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right)$$

sit $n = 1$ et functio Φ euanescat erit

$$z = \frac{1}{2a} PP - Py = x$$

at casus $n = 2$ dat

$$z = \frac{2}{3a} P^3 - P^2 y = \tfrac{2}{3} axy + \tfrac{2}{3} P(2x + ayy), \text{ seu}$$

$$z = aay^3 + 3axy + (ayy + 2x)\sqrt{(aayy + 2ax)}.$$

Scholion.

311. Forma integralis inuenta sequenti modo simplicior effici potest: Ponatur

$$\int \frac{dP}{\sqrt{P}} F : P = \Pi : P \text{ erit}$$

$$F : P = \sqrt{P} . \Pi' : P$$

eritque omittendo postremum membrum

$$\Phi \left(\int \frac{dP}{\sqrt{P}} f' : P - 2y\sqrt{P}\right)$$

quod nulla reductione indiget

$$z = \int dP\left(\sqrt{P}.f':P - \tfrac{1}{2}\int \tfrac{dP}{\sqrt{P}} f':P\right)\Pi':P + \tfrac{1}{2}\Pi:P.\int \tfrac{dP}{\sqrt{P}} f':P - y\sqrt{P}.\Pi:P$$

at $\tfrac{1}{2}\Pi:P.\int \tfrac{dP}{\sqrt{P}} f':P = \int\left(\tfrac{1}{2}dP\,\Pi':P.\int \tfrac{dP}{\sqrt{P}} f':P + \tfrac{1}{2}\tfrac{dP}{\sqrt{P}}\Pi:P.f':P\right)$

vnde fit

$$z = \int \Pi':P.dP\sqrt{P}f':P + \tfrac{1}{2}\int \Pi:P.\tfrac{dP}{\sqrt{P}} f':P - y\sqrt{P}.\Pi:P.$$

Porro est

$$\int dP.\Pi':P.\sqrt{P}.f':P = \Pi:P.\sqrt{P}.f':P - \int \Pi:P\left(\tfrac{dP}{2\sqrt{P}} f':P + dP\sqrt{P}.f'':P\right)$$

ideoque

$$z = \Pi:P.\sqrt{P}.f':P - \int dP.\Pi:P.\sqrt{P}.f'':P - y\sqrt{P}.\Pi:P$$

statuatur porro

$\int dP.\Pi:P.\sqrt{P}.f'':P = \Theta:P.$ erit

$\Pi:P = \frac{\Theta':P}{\sqrt{P}.f'':P}$ et

$$z = \frac{\Theta':P}{f'':P}(f':P - y) - \Theta:P + \Phi\left(\int \tfrac{dP}{\sqrt{P}} f':P - 2y\sqrt{P}\right)$$

quae forma sine dubio multo est simplicior quam primo inuenta.

Problema 50.

312. Proposita aequatione

$$\left(\frac{ddz}{dy^2}\right) - PP\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dy}\right) + R\left(\frac{dz}{dx}\right) = 0$$

inuenire casus quantitatum P, Q, R, quibus integratio ope reductionis ante adhibitae succedit.

Solutio.

Solutio.

Introductis binis nouis variabilibus t et u, habebimus:

$$\begin{aligned}
0=&\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dt\,du}\right)+\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right)\\
&-PP\left(\frac{ddt}{dx^2}\right)-PP\left(\frac{ddu}{dx^2}\right)-PP\left(\frac{dt}{dx}\right)^2-2PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)-PP\left(\frac{du}{dx}\right)^2\\
&+Q\left(\frac{dt}{dy}\right)\quad+Q\left(\frac{du}{dy}\right)\\
&+R\left(\frac{dt}{dx}\right)\quad+R\left(\frac{du}{dx}\right).
\end{aligned}$$

Statuamus ergo vt ante

$$\left(\frac{dt}{dy}\right)=P\left(\frac{dt}{dx}\right) \text{ et } \left(\frac{du}{dy}\right)=-P\left(\frac{du}{dx}\right)$$

vnde fit

$$\left(\frac{ddt}{dx\,dy}\right)=P\left(\frac{ddt}{dx^2}\right)+\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right) \text{ et}$$

$$\left(\frac{ddt}{dy^2}\right)=PP\left(\frac{ddt}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)+\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right)$$

atque

$$\left(\frac{ddu}{dy^2}\right)=PP\left(\frac{ddu}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{du}{dx}\right)-\left(\frac{dP}{dy}\right)\left(\frac{du}{dx}\right)$$

et aequatio resoluenda erit:

$$\begin{aligned}
0=&\left(P\left(\frac{dP}{dx}\right)+\left(\frac{dP}{dy}\right)+PQ+R\right)\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)-4PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dt\,du}\right)\\
&+\left(P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)-PQ+R\right)\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right)
\end{aligned}$$

Iam euidens est integrationem institui posse, si alterutra formula $\left(\frac{dz}{dt}\right)$ vel $\left(\frac{dz}{du}\right)$ ex calculo abeat. Ponamus ergo esse

$$P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)-PQ+R=0 \text{ seu}$$

$$R=PQ+\left(\frac{dP}{dy}\right)-P\left(\frac{dP}{dx}\right)$$

 et

et aequatio resultans per $\left(\frac{dt}{dx}\right)$ diuisa fit

$$0 = 2\left(PQ + \left(\frac{dP}{dy}\right)\right)\left(\frac{dz}{dt}\right) - 4PP\left(\frac{du}{dx}\right)\left(\frac{ddz}{dt\,du}\right).$$

Fiat $\left(\frac{dz}{dt}\right) = v$, erit

$$\left(PQ + \left(\frac{dP}{dy}\right)\right)v - 2PP\left(\frac{du}{dx}\right)\left(\frac{dv}{du}\right) = 0$$

sumatur t constans, vt fiat

$$\frac{dv}{v} = \frac{\left(PQ + \left(\frac{dP}{dy}\right)\right)du}{2PP\left(\frac{du}{dx}\right)}$$

vbi necesse est, vt quantitates P, Q, $\left(\frac{dP}{dy}\right)$ et $\left(\frac{du}{dx}\right)$ per nouas variabiles t et u exprimantur. Has ergo primum definiri conuenit. Cum sit

$$\left(\frac{dt}{dy}\right) = P\left(\frac{dt}{dx}\right) \text{ et } \left(\frac{du}{dy}\right) = -P\left(\frac{du}{dx}\right)$$

erit

$$dt = \left(\frac{dt}{dx}\right)(dx + Pdy) \text{ et } du = \left(\frac{du}{dx}\right)(dx - Pdy)$$

sunt ergo $\left(\frac{dt}{dx}\right)$ et $\left(\frac{du}{dx}\right)$ factores integrabiles reddentes formulas $dx + Pdy$ et $dx - Pdy$, non enim opus est vt hinc valores t et u generalissime definiantur. Sint p et q tales multiplicatores, per x et y dati eritque

$$t = \int p(dx + Pdy) \text{ et } u = \int q(dx - Pdy)$$

vnde superior integratio fit

$$\frac{dv}{v} = \frac{\left(PQ + \left(\frac{dP}{dy}\right)\right)du}{2PPq}$$

in

in qua integratione quantitas $t = \int p(dx + P dy)$ conftans eft fpectanda. Seu ob $du = q(dx - P dy)$ erit

$$\frac{dv}{v} = \frac{(PQ + (\frac{dP}{dy}))(dx - P dy)}{2PP}.$$

Verum ob $dt = 0$ eft $dx = -P dy$, ita vt prodeat

$$\frac{dv}{v} = -\frac{dy}{P}(PQ + (\frac{dP}{dy}))$$

vbi ob t conftans, et datum per x et y, valor ipfius x per y et t expreffus fubftitui poteft, vt fola y variabilis infit et inuento integrali

$$-\int \frac{dy}{P}(PQ + (\frac{dP}{dy})) = lV$$

erit $v = V f : t = (\frac{dz}{dt})$.

Nunc ponatur u conftans erit

$$z = \int V dt f : t + F : u.$$

Conditio autem fub qua haec integratio locum habet, poftulat vt fit

$$R = PQ + (\frac{dP}{dy}) - P(\frac{dP}{dx}).$$

Coroll. 1.

313. Eodem modo aequatio propofita refolutionem admittet, fi fuerit

$$R = -PQ - (\frac{dP}{dy}) - P(\frac{dP}{dx});$$

manetque vt ante

$$t = \int p(dx + P dy) \text{ et } u = \int q(dx - P dy).$$

 Tum

Tum vero fit:

$$0 = -(PQ + (\tfrac{dP}{dy}))(\tfrac{dz}{du}) - 2PP(\tfrac{dt}{dx})(\tfrac{ddz}{dtdu})$$

quae posito $(\frac{dz}{du}) = v$, sumtoque u constante dat

$$\frac{dv}{v} = \frac{-(PQ + (\frac{dP}{dy}))dt}{2PP(\frac{dt}{dx})} = \frac{-(PQ + (\frac{dP}{dy}))(dx + Pdy)}{2PP}.$$

Coroll. 2.

314. Si porro habita ratione, quod

$$u = fq(dx - Pdy)$$

sit constans et $dx = Pdy$, ponatur

$$f - \frac{dy(PQ + (\frac{dP}{dy}))}{P} = lV, \text{ erit}$$

$$v = Vf : u = (\tfrac{dz}{du}),$$

vnde tandem sumendo iam

$$t = fp(dx + Pdy),$$

colligitur:

$$z = f V du f : u + F : t.$$

Exemplum 1.

315. *Si sumatur* $P = a$, *et* $R = aQ$, *quaecunque fuerit* Q *functio ipsarum* x *et* y, *integrare aequationem*:

$$(\tfrac{ddz}{dy^2}) - aa(\tfrac{ddz}{dx^2}) + Q(\tfrac{dz}{dy}) + aQ(\tfrac{dz}{dx}) = 0.$$

Cum

Cum hic ſit $P=a$, erit $p=1$; $q=1$ et $t=x+ay$ atque $u=x-ay$ vnde poſito $\left(\frac{dz}{dt}\right)=v$ fit

$$\frac{dv}{v}=\frac{aQdu}{2aa}=\frac{Qdu}{2a}.$$

Quoniam igitur eſt

$$x=\frac{t+u}{2} \text{ et } y=\frac{t-u}{2a},$$

his valoribus ſubſtitutis fit Q functio ipſarum t et u, ac ſpectata t vt conſtante erit

$$lv=\frac{1}{2a}\int Q\,du+lf:t \text{ ſeu}$$

$$\left(\frac{dz}{dt}\right)=e^{\frac{1}{2a}\int Q du}f:t$$

et ſumta iam u conſtante

$$z=\int e^{\frac{1}{2a}\int Q du}dt\,f:t+F:u.$$

Coroll. 1.

316. Si Q ſit conſtans $=2ab$ aequationis huius:

$$\left(\frac{ddz}{dy^2}\right)-aa\left(\frac{ddz}{dx^2}\right)+2ab\left(\frac{dz}{dy}\right)+2aab\left(\frac{dz}{dx}\right)=0.$$

integrale erit:

$$z=e^{bu}f:t+F:u=e^{b(x-ay)}f:(x+ay)+F:(x-ay)$$

ſiue

$$z=e^{b(x-ay)}(f:(x+ay)+F:(x-ay)).$$

Coroll. 2.

Coroll. 2.

317. Si $Q = \frac{a}{x}$ huius aequationis

$$\left(\frac{ddz}{dy^2}\right) - aa\left(\frac{ddz}{dx^2}\right) + \frac{a}{x}\left(\frac{dz}{dx}\right) + \frac{aa}{x}\left(\frac{dz}{dx}\right) = 0$$

integrale ob

$$\int Q\,du = \int \frac{a\,du}{x} = \int \frac{2a\,du}{t+u} = 2a\,l(t+u) \text{ erit}$$

$$z = \int (t+u)\,dt\,f:t + F:u = \int t\,dt\,f:t + u\int dt\,f:t + F:u.$$

Vel sit $f:t = \Pi'':t$ erit

$$\int dt\,f:t = \Pi':t \text{ et}$$

$$\int t\,dt\,f:t = \int t\,d.\Pi':t = t\,\Pi':t - \int dt.\Pi':t = t\,\Pi':t - \Pi:t$$

ergo

$$z = (t+u)\Pi':t - \Pi:t + F:u \text{ seu}$$

$$z = 2x\Pi':(x+ay) - \Pi:(x+ay) + F:(x-ay).$$

Exemplum 2.

318. *Sit* $P = \frac{x}{y}$, *et* $R = \frac{-x}{y}Q + \frac{x}{yy} - \frac{x}{yy} = \frac{-x}{y}Q$, *sumaturque* $Q = \frac{1}{x}$ *vt sit* $R = \frac{-1}{y}$, *et haec aequatio integrari debeat*

$$\left(\frac{ddz}{dy^2}\right) - \frac{xx}{yy}\left(\frac{ddz}{dx^2}\right) + \frac{1}{x}\left(\frac{dz}{dy}\right) - \frac{1}{y}\left(\frac{dz}{dx}\right) = 0.$$

Cum ergo sit

$$t = \int p\left(dx + \frac{x\,dy}{y}\right) \text{ et } u = \int q\left(dx - \frac{x\,dy}{y}\right)$$

sumatur $p = y$ et $q = \frac{1}{y}$ vt fiat $t = xy$ et $u = \frac{x}{y}$.

Posito

Posito nunc $\left(\frac{dz}{du}\right) = v$ sumtoque u constante ex Coroll. 1. fit:

$$\frac{dv}{v} = \frac{-\left(\frac{1}{y} - \frac{x}{yy}\right)dt}{\frac{2xx}{yy} \cdot y} = \frac{-(y-x)dt}{2xxy}.$$

Est vero $tu = xx$ hincque $x = \sqrt{tu}$ et $y = \sqrt{\frac{t}{u}}$, atque $2xxy = 2t\sqrt{tu}$ vnde fit

$$\frac{dv}{v} = \frac{\left(\sqrt{tu} - \sqrt{\frac{t}{u}}\right)dt}{2t\sqrt{tu}} = \frac{dt}{2t} - \frac{dt}{2tu},$$

et ob u constans

$$lv = \tfrac{1}{2}lt - \tfrac{1}{2u}lt, \text{ ergo}$$

$$\left(\frac{dz}{du}\right) = t^{\frac{1}{2} - \frac{1}{2u}} f:u.$$

Quare sumto iam t constante erit

$$z = t^{\frac{1}{2}} \int t^{-\frac{1}{2u}} du f:u + F:t.$$

Vel ponatur $-\frac{1}{2u} = s$, vt sit $s = -\frac{y}{2x}$, eritque

$$z = t^{\frac{1}{2}} \int t^s ds f:s + F:t.$$

In hac integratione $\int t^s ds f:s$ sola s est variabilis, ac demum integrali sumto restitui debet $t = xy$ et $s = \frac{-y}{2x}$. Ceterum patet functionem quamcunque ipsius xy particulariter satisfacere.

Problema 51.

319. Propofita aequatione generali

$$\left(\frac{ddz}{dy^2}\right)-2P\left(\frac{ddz}{dx\,dy}\right)+(PP-QQ)\left(\frac{ddz}{dx^2}\right)+R\left(\frac{dz}{dy}\right)$$
$$+S\left(\frac{dz}{dx}\right)+Tz+V=o$$

inuenire conditiones quantitatum P, Q, R, S, T, vt integratio ope reductionis adhibitae fuccedat.

Solutio.

Facta eadem fubftitutione introducendis binis nouis variabilibus t et u, aequatio noftra fequentem induet formam:

$$\left.\begin{array}{lllll}
V+Tz+\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right) & +\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right) & +\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right) & +2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dt\,du}\right) & +\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right) \\
\quad -2P\left(\frac{ddt}{dx\,dy}\right) & -2P\left(\frac{ddu}{dx\,dy}\right) & -2P\left(\frac{dt}{dx}\right)\left(\frac{dt}{dy}\right) & -2P\left(\frac{dt}{dx}\right)\left(\frac{du}{dy}\right) & -2P\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) \\
\quad +(PP-QQ)\left(\frac{ddt}{dx^2}\right) & +(PP-QQ)\left(\frac{ddu}{dx^2}\right) & +(PP-QQ)\left(\frac{dt}{dx}\right)^2 & -2P\left(\frac{du}{dx}\right)\left(\frac{dt}{dy}\right) & +(PP-QQ)\left(\frac{du}{dx}\right)^2 \\
\quad +R\left(\frac{dt}{dy}\right) & +R\left(\frac{du}{dy}\right) & & +2(PP-QQ)\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right) & \\
\quad +S\left(\frac{dt}{dx}\right) & +S\left(\frac{du}{dx}\right). & & &
\end{array}\right\}=o$$

Determinentur iam hae duae nouae variabiles t et u ita per x et y, vt formulae $\left(\frac{ddz}{dt^2}\right)$ et $\left(\frac{ddz}{du^2}\right)$ euanefcant: debebitque effe

$$\left(\frac{dt}{dy}\right)=(P+Q)\left(\frac{dt}{dx}\right) \text{ et } \left(\frac{du}{dy}\right)=(P-Q)\left(\frac{du}{dx}\right),$$

vnde patet has variabiles fequenti modo determinari:

$$t=\int p(dx+(P+Q)dy) \text{ et } u=\int q(dx+(P-Q)dy)$$

fumendo

sumendo p et q ita vt hae formulae integrationem admittant. Cum nunc sit

$$\left(\frac{ddt}{dxdy}\right)=(P+Q)\left(\frac{ddt}{dx^2}\right)+\left[\left(\frac{dP}{dx}\right)+\left(\frac{dQ}{dx}\right)\right]\left(\frac{dt}{dx}\right)$$

$$\left(\frac{ddt}{dy^2}\right)=(P+Q)^2\left(\frac{ddt}{dx^2}\right)+(P+Q)\left[\left(\frac{dP}{dx}\right)+\left(\frac{dQ}{dx}\right)\right]\left(\frac{dt}{dx}\right)$$
$$+\left[\left(\frac{dP}{dy}\right)+\left(\frac{dQ}{dy}\right)\right]\left(\frac{dt}{dx}\right)$$

$$\left(\frac{ddu}{dxdy}\right)=(P-Q)\left(\frac{ddu}{dx^2}\right)+\left[\left(\frac{dP}{dx}\right)-\left(\frac{dQ}{dx}\right)\right]\left(\frac{du}{dx}\right)$$

$$\left(\frac{ddu}{dy^2}\right)=(P-Q)^2\left(\frac{ddu}{dx^2}\right)+(P-Q)\left[\left(\frac{dP}{dx}\right)-\left(\frac{dQ}{dx}\right)\right]\left(\frac{du}{dx}\right)$$
$$+\left[\left(\frac{dP}{dy}\right)-\left(\frac{dQ}{dy}\right)\right]\left(\frac{du}{dx}\right).$$

Hinc reperitur formulae $2\left(\frac{ddz}{dtdu}\right)$ coefficiens: $-2QQ\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)$ termini $\left(\frac{dz}{dt}\right)$ coefficiens:

$$\left[-(P-Q)\left(\frac{dP+dQ}{dx}\right)+\left(\frac{dP+dQ}{dy}\right)+R(P+Q)+S\right]\left(\frac{dt}{dx}\right)$$

termini vero $\left(\frac{dz}{du}\right)$ coefficiens:

$$\left[-(P+Q)\left(\frac{dP-dQ}{dx}\right)+\left(\frac{dP-dQ}{dy}\right)+R(P-Q)+S\right]\left(\frac{du}{dx}\right).$$

Est vero $\left(\frac{dt}{dx}\right)=p$ et $\left(\frac{du}{dx}\right)=q$: vnde si breuitatis gratia vocetur:

$$S+R(P+Q)+\left(\frac{dP+dQ}{dy}\right)-(P-Q)\left(\frac{dP+dQ}{dx}\right)=M \text{ et}$$
$$S+R(P-Q)+\left(\frac{dP-dQ}{dy}\right)-(P+Q)\left(\frac{dP-dQ}{dx}\right)=N$$

aequatio nostra resoluenda erit

$$0=V+Tz+Mp\left(\frac{dz}{dt}\right)+Nq\left(\frac{dz}{du}\right)-4QQpq\left(\frac{ddz}{dtdu}\right)$$

seu vt cum formis supra §. §. 294 et 295. exhibitis comparari queat:

$$\left(\frac{ddz}{dtdu}\right)-\frac{M}{4QQq}\left(\frac{dz}{dt}\right)-\frac{N}{4QQp}\left(\frac{dz}{du}\right)-\frac{T}{4QQpq}z-\frac{V}{4QQpq}=0$$

 quae

quae si porro breuitatis gratia ponatur

$$\frac{M}{4QQq} = K \text{ et } \frac{N}{4QQp} = L$$

duplici casu integrationem admittit: altero si fuerit

$$-\frac{T}{4QQpq} = +KL - \left(\frac{dL}{du}\right) \text{ seu } T = 4QQpq\left(\frac{dL}{du}\right) - \frac{MN}{4QQ}$$

altero vero si fuerit

$$-\frac{T}{4QQpq} = KL - \left(\frac{dK}{dt}\right) \text{ seu } T = 4QQpq\left(\frac{dK}{dt}\right) - \frac{MN}{4QQ}.$$

Quoniam vero K et L per x et y dantur, formulae illae $\left(\frac{dK}{dt}\right)$ et $\left(\frac{dL}{du}\right)$ ita reduci possunt vt sit

$$\left(\frac{dK}{dt}\right) = \frac{Q-P}{2Qp}\left(\frac{dK}{dx}\right) + \frac{1}{2Qp}\left(\frac{dK}{dy}\right) \text{ et}$$

$$\left(\frac{dL}{du}\right) = \frac{P+Q}{2Qq}\left(\frac{dL}{dx}\right) - \frac{1}{2Qq}\left(\frac{dL}{dy}\right).$$

Quemadmodum autem ipsa integralia his casibus inveniri debeant, id quidem supra est declaratum: vnde superfluum foret calculos illos taediosos hic repetere: quouis enim casu oblato solutio inde peti poterit.

Scholion 1.

920. Quod ad hanc reductionem formularum attinet, ea sequenti modo instituitur: Cum sit in genere

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right),$$

ex formulis

$$dt = pdx + p(P+Q)dy \text{ et } du = qdx + q(P-Q)dy$$

erit $qdt - pdu = 2pqQdy$ seu $dy = \frac{qdt - pdu}{2Qpq}$

et $q(P-Q)dt - p(P+Q)du = -2Qpqdx$

seu $dx = \frac{p(P+Q)du - q(P-Q)dt}{2Qpq}$.

Quibus

Quibus valoribus substitutis obtinebitur:

$$dz = \left(\frac{(P+Q)\,du}{2Qq} - \frac{(P-Q)\,dt}{2Qp}\right)\left(\frac{dz}{dx}\right) + \left(\frac{dt}{2Qp} - \frac{du}{2Qq}\right)\left(\frac{dz}{dy}\right)$$

ita vt dz per differentialia dt et du exprimatur. Posito ergo u constante et $du=0$ erit

$$\left(\frac{dz}{dt}\right) = \frac{Q-P}{2Qp}\left(\frac{dz}{dx}\right) + \frac{1}{2Qp}\left(\frac{dz}{dy}\right)$$

at posito t constante et $dt=0$ erit

$$\left(\frac{dz}{du}\right) = \frac{P+Q}{2Qq}\left(\frac{dz}{dx}\right) - \frac{1}{2Qq}\left(\frac{dz}{dy}\right).$$

Scholion 2.

321. Methodus igitur hoc capite tradita in hoc consistit vt huiusmodi aequationes ope introductionis binarum nouarum variabilium t et u ad hanc formam reducantur

$$\left(\frac{ddz}{dt\,du}\right) + P\left(\frac{dz}{dt}\right) + Q\left(\frac{dz}{du}\right) + Rz + S = 0$$

de qua in praecedente capite vidimus, quibusnam casibus ea integrari queat: Iisdem igitur quoque casibus omnes aequationes, quae ad talem formam se reduci patiuntur, integrationem admittent. Est vero eiusdem formae casus quidam maxime singularis, cuius integratio absolui potest, vnde denuo infinita multitudo aliarum aequationum, quae quidem eo reduci queant, oritur integrationem pariter admittentium. Quem propterea casum sequenti capite diligentius euoluamus.

CAPVT IV.

ALIA METHODVS PECVLIARIS HVIVSMODI AEQVATIONES INTEGRANDI.

Problema 52.

322.

Si aequatio propofita hanc habuerit formam:
$$(x+y)^2\left(\frac{ddz}{dx\,dy}\right)+m(x+y)\left(\frac{dz}{dx}\right)+m(x+y)\left(\frac{dz}{dy}\right)+nz=0$$
eius integrale completum inueftigare.

Solutio.

Cum hic binae variabiles x et y aequaliter infint ponatur primo
$$z=A(x+y)^\lambda f:x+B(x+y)^{\lambda+1}f':x+C(x+y)^{\lambda+2}f'':x$$
$$+D(x+y)^{\lambda+3}f''':x\text{ etc.}$$
vbi pro faciliori fubftitutione notetur, pofito $v=(x+y)^\mu F:x$ fore
$$\left(\frac{dv}{dx}\right)=\mu(x+y)^{\mu-1}F:x+(x+y)^\mu F':x$$
$$\left(\frac{dv}{dy}\right)=\mu(x+y)^{\mu-1}F:x \text{ et}$$
$$\left(\frac{ddv}{dx\,dy}\right)=\mu(\mu-1)(x+y)^{\mu-2}F:x+\mu(x+y)^{\mu-1}F':x.$$

Facta

Facta ergo substitutione obtinebimus hanc aequationem

$$
\begin{array}{llll}
0 = nA(x+y)^{\lambda} f : x & + nB(x+y)^{\lambda+1} f' : x & + nC(x+y)^{\lambda+2} f'' : x & + \text{etc} \\
+2m\lambda A & +mA & +mB & \\
+\lambda(\lambda-1)A & +2m(\lambda+1)B & +2m(\lambda+2)C & \\
 & +\lambda A & +(\lambda+1)B & \\
 & +(\lambda+1)\lambda B & +(\lambda+2)(\lambda+1)C &
\end{array}
$$

vbi totum negotium ad coefficientium A, B, C, D etc. determinationem reuocatur; facile autem erat praevidere, forma superiori assumta potestates ipsius $(x+y)$ in singulis membris pares esse prodituras: Fieri igitur necesse est

$$n+2m\lambda+\lambda\lambda-\lambda=0$$

$$(n+2m\lambda+2m+\lambda\lambda+\lambda)B+(m+\lambda)A=0$$

$$(n+2m\lambda+4m+\lambda\lambda+3\lambda+2)C+(m+\lambda+1)B=0$$

$$(n+2m\lambda+6m+\lambda\lambda+5\lambda+6)D+(m+\lambda+2)C=0$$

etc.

quae determinationes ope primae $n+2m\lambda+\lambda\lambda-\lambda=0$ ita commodius exprimuntur:

$$B=-\frac{(m+\lambda)A}{2(m+\lambda)} \qquad F=-\frac{(m+\lambda+4)E}{5(2m+2\lambda+4)}$$

$$C=-\frac{(m+\lambda+1)B}{2(2m+2\lambda+1)} \qquad G=-\frac{(m+\lambda+5)F}{6(2m+2\lambda+5)}$$

$$D=-\frac{(m+\lambda+2)C}{3(2m+2\lambda+2)} \qquad H=-\frac{(m+\lambda+6)G}{7(2m+2\lambda+6)}$$

$$E=-\frac{(m+\lambda+3)D}{4(2m+2\lambda+3)} \qquad \text{etc.}$$

vnde

vnde lex progressionis est manifesta. At pro exponente λ duplicem eruimus valorem

$$\lambda = \tfrac{1}{2} - m \pm \sqrt{(\tfrac{1}{4} - m - n + mm)}$$

quorum vtrumque aeque pro λ accipere licet. Hic autem praecipue notandi sunt casus, quibus series assumta abrumpitur, quod fit, quoties $m + \lambda + i = 0$ denotante i numerum quemcunque integrum positiuum cyphra non exclusa. Hoc ergo euenit quoties fuerit

$$\tfrac{1}{2} + i \pm \sqrt{(\tfrac{1}{4} - m - n + mm)} = 0$$

id quod fieri nequit nisi $\tfrac{1}{4} - m - n + mm$ fuerit quadratum. Inuenta autem huiusmodi serie siue finita siue in infinitum excurrente, alia similis pro functionibus ipsius y reperitur, vnde valor ipsius z ita reperietur expressus

$$\begin{aligned} z = {} & A(x+y)^{\lambda}(f:x + F:y) + B(x+y)^{\lambda+1}(f':x + F':y) \\ & + C(x+y)^{\lambda+2}(f'':x + F'':y) + D(x+y)^{\lambda+3}(f''':x + F''':y) \\ & + E(x+y)^{\lambda+4}(f^{\mathrm{IV}}:x + F^{\mathrm{IV}}:y) + F(x+y)^{\lambda+5}(f^{\mathrm{V}}:x + F^{\mathrm{V}}:y) \\ & \text{etc.} \end{aligned}$$

vbi cum binae functiones arbitrariae adsint, id certum est signum, hanc formam esse integrale completum aequationis propositae.

Coroll. 1.

323. Si fuerit $\lambda = -m$, hoc est $n - mm + m = 0$ seu $n = mm - m$, integrale ex vnico membro constabit

ſtabit ob $B=0$, eritque integrale

$$z=A(x+y)^{-m}(f:x+F:y).$$

Coroll. 2.

324. Integrale autem duo membra continebit, ſi $\lambda=-m-1$ vel $n=mm-m-2=(m+1)(m-2)$; tum erit $B=-\frac{1}{2}A$ et integrale erit

$$z=(x+y)^{-m-1}(f:x+F:y)-\tfrac{1}{2}(x+y)^{-m}(f':x+F':y).$$

Coroll. 3.

325. Integrale tribus terminis conſtabit, ſi $\lambda=-m-2$ vel $n=(m+2)(m-3)$; tum erit

$$B=-\tfrac{1}{2}A;\ \text{et}\ C=-\tfrac{1}{6}B=+\tfrac{1}{12}A;$$

integrale vero

$$z=(x+y)^{-m-2}(f:x+F:y)-\tfrac{1}{2}(x+y)^{-m-1}(f':x+F':y)$$
$$+\tfrac{1}{12}(x+y)^{-m}(f'':x+F'':y).$$

Coroll. 4

326. Ex quatuor autem membris integrale conſtabit, ſi fuerit $\lambda=-m-3$, ſeu $n=(m+3)(m-4)$; tum autem erit

$$B=-\tfrac{1}{2}A;\ C=-\tfrac{1}{5}B=+\tfrac{1}{10}A;\ D=-\tfrac{1}{12}C=-\tfrac{1}{120}A$$

et integrale:

$$z=(x+y)^{-m-3}(f:x+F:y)-\tfrac{1}{2}(x+y)^{-m-2}(f':x+F':y)$$
$$+\tfrac{1}{10}(x+y)^{-m-1}(f'':x+F'':y)-\tfrac{1}{120}(x+y)^{-m}(f''':x+F''':y).$$

Scholion.

327. Quod si in genere ponamus $\lambda+m=-i$ erit $n=(m+i)(m-i-1)$, tum vero

$$B=-\tfrac{1}{2}A;\quad C=-\frac{(i-1)B}{2(2i-1)};\quad D=-\frac{(i-2)C}{3(2i-2)};\quad E=-\frac{(i-3)D}{4(2i-3)}$$

vnde fit omnes ad primum reducendo:

$$B=-\tfrac{1}{2}A;\quad C=\frac{(i-1)}{2\cdot2(2i-1)}A;\quad D=\frac{-(i-1)(i-2)}{2\cdot2\cdot2\cdot3(2i-1)(2i-2)}A;$$

$$E=\frac{+(i-2)(i-3)}{2\cdot2\cdot2\cdot3\cdot4(2i-1)(2i-3)}A;\quad F=\frac{-(i-3)(i-4)}{2^4\cdot3\cdot4\cdot5(2i-3)(2i-4)}A \text{ etc.}$$

qui ita se habent

	A	B	C	D	E	F
$i=1$	1	$-\frac{1}{2}$	0	0	0	0
$i=2$	1	$-\frac{1}{2}$	$\frac{1}{12}$	0	0	0
$i=3$	1	$-\frac{1}{2}$	$\frac{2}{20}$	$-\frac{1}{120}$	0	0
$i=4$	1	$-\frac{1}{2}$	$\frac{3}{28}$	$-\frac{2}{7\cdot24}$	$\frac{2\cdot1}{96\cdot7\cdot5}$	0
$i=5$	1	$-\frac{1}{2}$	$\frac{4}{36}$	$-\frac{3}{9\cdot24}$	$\frac{3\cdot2}{96\cdot9\cdot7}$	$\frac{2\cdot1}{960\cdot9\cdot7}$
$i=6$	1	$-\frac{1}{2}$	$\frac{4}{44}$	$-\frac{4}{11\cdot24}$	$\frac{4\cdot3}{96\cdot11\cdot9}$	$\frac{3\cdot2}{960\cdot11\cdot9}$

Ita huius aequationis:

$$\left(\frac{ddz}{dx\,dy}\right)+\frac{m}{x+y}\left(\frac{dz}{dx}\right)+\frac{m}{x+y}\left(\frac{dz}{dy}\right)+\frac{(m+i)(m-i-1)}{(x+y)^2}z=0$$

integrale completum erit:

$$\begin{aligned}
z=&+(x+y)^{-m-i}(f:x+F:y)\\
&-\frac{i}{2i}(x+y)^{-m-i+1}(f':x+F':y)\\
&+\frac{i(i-1)}{2i\cdot2(2i-1)}(x+y)^{-m-i+2}(f'':x+F'':y)\\
&-\frac{i(i-1)(i-2)}{2i\cdot2(2i-1)\cdot3(2i-2)}(x+y)^{-m-i+3}(f''':x+F''':y)\\
&+\frac{i(i-1)(i-2)(i-3)}{2i\cdot2(2i-1)\cdot3(2i-2)\cdot4(2i-3)}(x+y)^{-m-i+4}(f^{IV}:x+F^{IV}:y)\\
&-\frac{i(i-1)(i-2)(i-3)(i-4)}{2i\cdot2(2i-1)\cdot3(2i-2)\cdot4(2i-3)\cdot5(2i-4)}(x+y)^{-m-i+5}(f^{V}:x+F^{V}:y)\\
&\text{etc.}
\end{aligned}$$

quae

quae forma quoties i fuerit numerus integer positiuus, finito constat terminorum numero: secus autem in infinitum excurrit. Imprimis autem ista integratio hoc habet singulare, quod non solum ipsas functiones arbitrarias $f:x$ et $F:y$ complectatur, sed etiam earum formulas differentiales.

Exemplum.

328. *Si occurrat ista aequatio*

$$\left(\frac{ddz}{dx\,dy}\right)+\frac{m}{x+y}\left(\frac{dz}{dx}\right)+\frac{m}{x+y}\left(\frac{dz}{dy}\right)=0$$

definire casus, quibus eius integrale per formam finitam exhiberi potest.

Cum hic sit $n=(m+i)(m-i-1)=0$ sumendo pro i numeros integros positiuos, duo ordines habebuntur casuum, quibus integratio succedit, alter quo est $m=-i$, alter quo $m=i+1$ ita vt in genere integratio finita locum habeat, quoties m fuerit numerus integer siue positiuus siue negatiuus. Primo ergo si sit $m=-i$ erit

$$\begin{aligned}z=1(f:x+F:y)&-\frac{i}{2i}(x+y)(f':x+F':y)\\
&+\frac{1}{2}\cdot\frac{i(i-1)}{2i(2i-1)}(x+y)^2(f'':x+F'':y)\\
&-\frac{1}{6}\cdot\frac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3(f''':x+F''':y)\\
&+\frac{1}{24}\cdot\frac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4(f^{IV}:x+F^{IV}:y).\end{aligned}$$

etc.

Deinde si sit $m = i + 1$ erit

$$(x+y)^{2i+1} z = 1\,(f:x + F:y) - \tfrac{i}{2i}(x+y)(f':x + F':y)$$
$$+ \tfrac{1}{2} \cdot \tfrac{i(i-1)}{2i(2i-1)}(x+y)^2 (f'':x + F'':y)$$
$$- \tfrac{1}{6} \cdot \tfrac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3 (f''':x + F''':y)$$
$$+ \tfrac{1}{24} \cdot \tfrac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4 (f^{IV}:x + F^{IV}:y)$$

etc.

vtrinque scilicet eadem habetur expressio, cui casu priori ipsa quantitas z, posteriori quantitas $(x+y)^{2i+1} z$ aequatur. Ad singulos hos casus distinctius euolvendos ponamus:

$$A = (f:x + F:y)$$
$$B = (f:x + F:y) - \tfrac{1}{2}(x+y)(f':x + F':y)$$
$$C = (f:x + F:y) - \tfrac{2}{4}(x+y)(f':x + F':y) + \tfrac{1}{4 \cdot 3}(x+y)^2 (f'':x + F'':y)$$
$$D = (f:x + F:y) - \tfrac{3}{6}(x+y)(f':x + F':y) + \tfrac{3}{6 \cdot 5}(x+y)^2 (f'':x + F'':y)$$
$$- \tfrac{1}{6 \cdot 5 \cdot 4}(x+y)^3 (f''':x + F''':y)$$

etc.

vel posito breuitatis gratia:

$$\mathfrak{A} = f:x + F:y\,;$$
$$\mathfrak{B} = (x+y)(f':x + F':y)\,;$$
$$\mathfrak{C} = (x+y)^2 (f'':x + F'':y)\,;$$
$$\mathfrak{D} = (x+y)^3 (f''':x + F''':y)\,;$$
$$\mathfrak{E} = (x+y)^4 (f^{IV}:x + F^{IV}:y)\,;$$

etc.

fit

fit

$A = \mathfrak{A}$

$B = \mathfrak{A} - \frac{1}{2}\mathfrak{B}$

$C = \mathfrak{A} - \frac{2}{4}\mathfrak{B} + \frac{1}{4 \cdot 3}\mathfrak{C}$

$D = \mathfrak{A} - \frac{3}{6}\mathfrak{B} + \frac{3}{6 \cdot 5}\mathfrak{C} - \frac{1}{6 \cdot 5 \cdot 4}\mathfrak{D}$

$E = \mathfrak{A} - \frac{4}{8}\mathfrak{B} + \frac{6}{8 \cdot 7}\mathfrak{C} - \frac{4}{8 \cdot 7 \cdot 6}\mathfrak{D} + \frac{1}{8 \cdot 7 \cdot 6 \cdot 5}\mathfrak{E}$

$F = \mathfrak{A} - \frac{5}{10}\mathfrak{B} + \frac{10}{10 \cdot 9}\mathfrak{C} - \frac{10}{10 \cdot 9 \cdot 8}\mathfrak{D} + \frac{5}{10 \cdot 9 \cdot 8 \cdot 7}\mathfrak{E} - \frac{1}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}\mathfrak{F}$

$$G = \mathfrak{A} - \frac{6}{12}\mathfrak{B} + \frac{15}{12 \cdot 11}\mathfrak{C} - \frac{20}{12 \cdot 11 \cdot 10}\mathfrak{D} + \frac{15}{12 \cdot 11 \cdot 10 \cdot 9}\mathfrak{E} - \frac{6}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}\mathfrak{F} + \frac{1}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}\mathfrak{G}$$

etc.

Quibus valoribus inuentis erit, pro duplici ordine:

fi	erit	fi	erit
$m = 0$;	$z = A$	$m = 1$;	$(x+y)\, z = A$
$m = -1$;	$z = B$	$m = 2$;	$(x+y)^3 z = B$
$m = -2$;	$z = C$	$m = 3$;	$(x+y)^5 z = C$
$m = -3$;	$z = D$	$m = 4$;	$(x+y)^7 z = D$
$m = -4$;	$z = E$	$m = 5$;	$(x+y)^9 z = E$
$m = -5$;	$z = F$	$m = 6$;	$(x+y)^{11} z = F$
$m = -6$;	$z = G$.	$m = 7$;	$(x+y)^{13} z = G$
etc.		etc.	

Scholion.

329. Si pro i fumatur numerus negatiuus, expreffio in infinitum excurrit. Sit enim $i = -k$,

et ex formula prima erit $m=k$ ideoque

$$z=\mathfrak{A}-\frac{k}{2k}\mathfrak{B}+\frac{1}{2}\cdot\frac{k(k+1)}{2k(2k+1)}\mathfrak{C}-\frac{1}{6}\cdot\frac{k(k+1)(k+2)}{2k(2k+1)(2k+2)}\mathfrak{D}+\text{etc.}$$

in infinitum. Pro eodem autem casu $m=k$ altera forma ob $i=k-1$ dat

$$(x+y)^{2k-1}z=\mathfrak{A}-\frac{(k-1)}{2k-2}\mathfrak{B}+\frac{1}{2}\cdot\frac{(k-1)(k-2)}{(2k-2)(2k-3)}\mathfrak{C}$$
$$-\frac{1}{6}\cdot\frac{(k-1)(k-2)(k-3)}{(2k-2)(2k-3)(2k-4)}\mathfrak{D}+\text{etc.}$$

quae autem formae non absolute aequales sunt censendae sed in altera functiones $f:x$ et $F:y$ alias formas habebunt, vt nihilominus ambae aeque satisfaciant. Casu quidem $k=\frac{1}{2}$, ambae conueniunt perfecte: ponamus autem $k=0$ vt prior det

$$z=\mathfrak{A}=f:x+F:y\,,$$

at posterior praebet

$$\frac{z}{x+y}=\mathfrak{A}-\tfrac{1}{2}\mathfrak{B}+\tfrac{1}{6}\mathfrak{C}-\tfrac{1}{24}\mathfrak{D}+\tfrac{1}{120}\mathfrak{E}-\text{etc.}$$

Quarum consensus vt appareat sit, in hac posteriori

$$f:x=ax^3 \text{ et } F:y=by^2 \text{ erit:}$$

$$\mathfrak{A}=ax^3+by^2;\ \mathfrak{B}=(x+y)(3axx+2by);$$
$$\mathfrak{C}=(x+y)^2(6ax+2b);\ \mathfrak{D}=(x+y)^3 6a$$

at reliquae partes euanescunt. Obtinebimus ergo ex posteriori

$$z=(x+y)(ax^3+by^2)-\tfrac{1}{2}(x+y)^2(3axx+2by)$$
$$+\tfrac{1}{3}(x+y)^3(3ax+b)-\tfrac{1}{4}(x+y)^4 a$$

quae euoluta praebet

$$\tfrac{1}{4}ax^4+\tfrac{1}{4}ay^4+\tfrac{1}{3}bx^3+\tfrac{1}{3}by^3=z\,,$$

quae

quae forma vtique in priori $z = f:x + F:y$ continetur. Confenfus ergo binarum illarum formarum generalium eo magis eft notatu dignus.

Problema 53.

330. Inuenire cafus quibus haec aequatio generalis:

$$\left(\frac{ddz}{dy^2}\right) - QQ\left(\frac{ddz}{dx^2}\right) + R\left(\frac{dz}{dy}\right) + S\left(\frac{dz}{dx}\right) + Tz = 0$$

ad formam praecedentem reduci, ideoque iisdem cafibus integrari poteft.

Solutio.

Introducendo binas nouas variabiles t et u, vt fit quemadmodum reductio §. 319. adhibita, vbi $P = 0$ et $V = 0$, declarat:

$$t = \int p(dx + Qdy) \text{ et } u = \int q(dx - Qdy)$$

fi ponamus ad abbreuiandum:

$$M = S + QR + \left(\frac{dQ}{dy}\right) + Q\left(\frac{dQ}{dx}\right)$$

$$N = S - QR - \left(\frac{dQ}{dy}\right) + Q\left(\frac{dQ}{dx}\right)$$

prodibit haec aequatio:

$$\left(\frac{ddz}{dtdu}\right) - \frac{M}{4QQq}\left(\frac{dz}{dt}\right) - \frac{N}{4QQp}\left(\frac{dz}{du}\right) - \frac{T}{4QQpq}z = 0$$

quam ergo ad hanc formam reuocari oportet:

$$\left(\frac{ddz}{dtdu}\right) + \frac{m}{t+u}\left(\frac{dz}{dt}\right) + \frac{m}{t+u}\left(\frac{dz}{du}\right) + \frac{n}{(t+u)^2}z = 0$$

cuius

cuius casus integrabilitatis ante designauimus, scilicet quoties fuerit $n=(m+i)(m-i-1)$, denotante i numerum integrum quemcunque positiuum cyphra non exclusa. Ad hoc ergo necesse est vt fiat:

$$M=\frac{-4mQQq}{t+u};\ N=\frac{-4mQQp}{t+u} \text{ et } T=\frac{-4nQQpq}{(t+u)^2}.$$

Quia autem hic integrabilitatis formularum t et u ratio haberi debet, sumamus $Q=\frac{\Phi':y}{\pi':x}$, sitque

$$p=a\pi':x \text{ et } q=b\pi':x$$

eritque

$$t=a\pi:x+a\Phi:y \text{ et } u=b\pi:x-b\Phi:y.$$

Hinc fit:

$$M+N=2S+2Q\left(\frac{dQ}{dx}\right)=\frac{-4m(a+b)QQ\pi':x}{t+u} \text{ et}$$

$$M-N=2QR+2\left(\frac{dQ}{dy}\right)=\frac{4m(a-b)QQ\pi':x}{t+u}$$

ideoque

$$R=\frac{2m(a-b)Q\pi':x}{t+u}-\frac{1}{Q}\left(\frac{dQ}{dy}\right)$$

$$S=\frac{-2m(a+b)QQ\pi':x}{t+u}-Q\left(\frac{dQ}{dx}\right) \text{ et}$$

$$T=\frac{-4nabQQ\pi':x.\pi':x}{(t+u)^2}=\frac{-4nab\Phi':y.\Phi':y}{(t+u)^2}$$

ob $Q=\frac{\Phi':y}{\pi':x}$; vnde est

$$\left(\frac{dQ}{dy}\right)=\frac{\Phi'':y}{\pi':x} \text{ et } \left(\frac{dQ}{dx}\right)=\frac{-\pi'':x.\Phi':y}{\pi':x.\pi':x} \text{ et}$$

$$t+u=(a+b)\pi:x+(a-b)\Phi:y.$$

Ideoque habebimus:

$$R=\frac{2m(a-b)\Phi':y}{t+u}-\frac{\Phi'':y}{\Phi':y} \text{ et}$$

$$\frac{S}{QQ}=\frac{-2m(a+b)\pi':x}{t+u}+\frac{\pi'':x}{\pi':x}.$$

Qua

Quo aequatio fiat ſimplicior, duo caſus praecipue ſunt conſiderandi, alter vbi $b=a$, alter $b=-a$. Priori eſt $t+u=2a\pi:x$ et aequatio noſtra erit

$$\left(\frac{ddz}{dy^2}\right)-\frac{\Phi':y.\Phi':y}{\pi':x.\pi':x}\left(\frac{ddz}{dx^2}\right)-\frac{\Phi'':y}{\Phi':y}\left(\frac{dz}{dy}\right)+\left(\frac{\Phi':y}{\pi':x}\right)^2\left(\frac{\pi'':x}{\pi':x}-\frac{2m\pi':x}{\pi:x}\right)\left(\frac{dz}{dx}\right)$$
$$-\frac{n\Phi':y.\Phi':y}{\pi x.\pi x}z=0.$$

Altero vero caſu $b=-a$ fit $t+u=2a\Phi:y$ et

$$\left(\frac{ddz}{dy^2}\right)-\left(\frac{\Phi':y}{\pi':x}\right)^2\left(\frac{ddz}{dx^2}\right)+\left(\frac{2m\Phi':y}{\Phi:y}-\frac{\Phi'':y}{\Phi':y}\right)\left(\frac{dz}{dy}\right)+\left(\frac{\Phi':y}{\pi':x}\right)^2.\frac{\pi'':x}{\pi':x}\left(\frac{dz}{dx}\right)$$
$$+\frac{n\Phi':y.\Phi':y}{\Phi:y.\Phi:y}z=0$$

quae ambae aequationes integrationem admittunt caſibus $n=(m+i)(m-i-1)$.

Coroll. 1.

331. Aequationes poſtremo inuentae a ſe inuicem non differunt, niſi quod binae variabiles x et y inuicem permutantur vnde ſufficit alterutram ſolam conſideraſſe. Prior autem transformatur ponendo

$$t=\pi:x+\Phi:y \text{ et } u=\pi:x-\Phi:y;$$

poſterior vero ponendo

$$t=\pi:x+\Phi:y \text{ et } u=\Phi:y-\pi:x.$$

Coroll. 2.

332. Hae aequationes etiam ſequenti forma magis perſpicua repraeſentari poſſunt, prior quidem

$$\frac{1}{(\Phi':y)^2}\left(\frac{ddz}{dy^2}\right)-\frac{1}{(\pi':x)^2}\left(\frac{ddz}{dx^2}\right)-\frac{\Phi'':y}{(\Phi':y)^3}\left(\frac{dz}{dy}\right)+\left(\frac{\pi'':x}{(\pi':x)^3}-\frac{2m}{\pi:x.\pi':x}\right)\left(\frac{dz}{dx}\right)$$
$$-\frac{n}{(\pi:x)^2}z=0$$

et

et posterior

$$\frac{1}{(\Phi':y)^2}\left(\frac{ddz}{dy^2}\right)-\frac{1}{(\pi':x)^2}\left(\frac{ddz}{dx^2}\right)+\left(\frac{2m}{\Phi:y.\Phi':y}-\frac{\Phi'':y}{(\Phi':y)^3}\right)\left(\frac{dz}{dy}\right)+\frac{\pi'':x}{(\pi':x)^3}\left(\frac{dz}{dx}\right)$$
$$+\frac{n}{(\Phi:y)^2}z=0.$$

Casus 1.

333. Ponamus $\pi':x=a$, et $\Phi':y=b$, erit $\pi:x=ax$ et $\Phi:y=by$ tum vero $\pi'':x=0$ et $\Phi'':y=0$; vnde forma prior prodibit:

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}\left(\frac{ddz}{dx^2}\right)-\frac{2m}{aax}\left(\frac{dz}{dx}\right)-\frac{n}{aaxx}z=0$$

quae reducitur ad formam supra resolutam ponendo

$$t=ax+by \text{ et } u=ax-by.$$

Posterior vero forma est

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{bby}\left(\frac{dz}{dy}\right)+\frac{n}{bbyy}z=0$$

quae reducitur ad formam supra resolutam ponendo

$$t=ax+by \text{ et } u=by-ax$$

vtraque autem est integrabilis casu

$$n=(m+i)(m-i-1).$$

Reductione enim ad variabiles t et u facta oritur haec aequatio

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)+\frac{n}{(t+u)^2}z=0.$$

Coroll. 1.

Coroll. 1.

334. Si ſumatur $n = 0$, hae ambae aequationes:

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right) - \left(\frac{ddz}{dx^2}\right) - \frac{2m}{x}\left(\frac{dz}{dx}\right) = 0 \text{ et}$$

$$\left(\frac{ddz}{dy^2}\right) - \frac{bb}{aa}\left(\frac{ddz}{dx^2}\right) + \frac{2m}{y}\left(\frac{dz}{dy}\right) = 0$$

ſunt integrabiles, quoties m fuerit numerus integer, ideoque $2m$ numerus par.

Coroll. 2.

335. En ergo aequationes ob ſimplicitatem notatu dignas, ex tribus tantum terminis conſtantes, quae infinitis caſibus integrationem admittunt. Integrale autem quouis caſu facile exhibetur ex ex §. 328, ſi modo ibi loco x et y ſcribatur t et u.

Caſus 2.

336. Sit $\pi' : x = ax^\mu$ et $\Phi' : y = b$, erit

$$\pi : x = \frac{1}{\mu+1} ax^{\mu+1} \text{ et } \Phi : y = by$$

tum vero

$$\pi'' : x = \mu ax^{\mu-1} \text{ et } \Phi'' : y = 0.$$

Vnde forma prior prouenit

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right) - \frac{1}{aax^{2\mu}}\left(\frac{ddx}{dx^2}\right) + \frac{\mu - 2m\mu - 2m}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right) - \frac{n(\mu+1)^2}{aax^{2\mu+2}} z = 0$$

 quae

quae reducitur ad formam supra resolutam ponendo

$$t = \frac{1}{\mu+1} a x^{\mu+1} + by \quad \text{et} \quad u = \frac{1}{\mu+1} a x^{\mu+1} - by.$$

Posterior vero forma fit

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right) - \frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right) + \frac{2m}{bby}\left(\frac{dz}{dy}\right) + \frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)$$

$$+ \frac{n}{bbyy} z = 0$$

cuius reductio absoluitur ponendo:

$$t = \frac{1}{\mu+1} a x^{\mu+1} + by \quad \text{et} \quad u = by - \frac{1}{\mu+1} a x^{\mu+1}.$$

Haeque ambae aequationes integrationem admittunt, quoties fuerit $n = (m+i)(m-i-1)$.

Coroll. 1.

337. Ex priori forma casus maxime notabilis existit, si capiatur $m = \frac{\mu}{2\mu+2}$, et $n = 0$, tum enim erit

$$\frac{aa}{bb} x^{2\mu}\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right),$$

quae est integrabilis, quoties $\frac{\mu}{2\mu+2}$ fuerit numerus integer m siue positiuus siue negatiuus.

Coroll. 2.

338. Vel cum sit $\mu = \frac{-2m}{2m-1}$, haec aequatio

$$\frac{aa}{bb} x^{\frac{-4m}{2m-1}}\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) \text{ seu } \left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)$$

erit

erit integrabilis, quoties m fuerit numerus integer siue positiuus siue negatiuus, reductio autem fit ponendo

$$t = -(2m-1)\, a\, x^{\frac{-1}{2m-1}} + by \text{ et}$$

$$u = -(2m-1)\, a\, x^{\frac{-1}{2m-1}} - by.$$

Casus 3.

339. Sit $\pi' : x = ax^{\mu}$ et $\Phi' : y = by^{\nu}$, erit

$$\pi : x = \frac{1}{\mu+1} ax^{\mu+1} \text{ et } \Phi : y = \frac{1}{\nu+1} by^{\nu+1},$$

tum vero

$$\pi'' : x = \mu a x^{\mu-1} \text{ et } \Phi'' : y = \nu b y^{\nu-1}.$$

Hinc prior forma resultat:

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right) - \frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right) - \frac{\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)$$
$$+ \frac{\mu - 2m\mu - 2m}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right) - \frac{n(\mu+1)^2}{aax^{2\mu+2}} z = 0$$

quae reducitur ponendo

$$t = \frac{1}{\mu+1} ax^{\mu+1} + \frac{1}{\nu+1} by^{\nu+1} \text{ et}$$
$$u = \frac{1}{\mu+1} ax^{\mu+1} - \frac{1}{\nu+1} by^{\nu+1}.$$

Posterior vero forma euadit

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right) - \frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right) + \frac{2m\nu + 2m - \nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)$$
$$+ \frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right) + \frac{n(\nu+1)^2}{bby^{2\nu+2}} z = 0$$

 cuius

cuius reductio fit hac substitutione

$$t = \frac{1}{\mu+1} a x^{\mu+1} + \frac{1}{\nu+1} b y^{\nu+1} \quad \text{et}$$

$$u = \frac{-1}{\mu+1} a x^{\mu+1} + \frac{1}{\nu+1} b y^{\nu+1}.$$

Vel cum hic tantum ratio inter a et b in computum ingrediatur, pro priori poni poterit:

$$t = \tfrac{1}{2} x^{\mu+1} + \frac{(\mu+1)b}{2(\nu+1)a} y^{\nu+1} \quad \text{et}$$

$$u = \tfrac{1}{2} x^{\mu+1} - \frac{(\mu+1)b}{2(\nu+1)a} y^{\nu+1}$$

vt fiat $t+u = x^{\mu+1}$ quo expressio integralis fiat simplicior.

Coroll. 1.

340. Si ponatur in forma priori $\mu = \frac{-2m}{2m-1}$, minuetur ea vno termino fietque:

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right) - \frac{1}{aa} x^{\frac{4m}{2m-1}} \left(\frac{ddz}{dx^2}\right) - \frac{\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)$$

$$- \frac{n}{(2m-1)^2 aa} x^{\frac{2}{2m-1}} z = 0.$$

Statuatur $a=b$ et capiatur quoque $\nu = \frac{-2m}{2m-1}$, vt prodeat

$$y^{\frac{4m}{2m-1}}\left(\frac{ddz}{dy^2}\right) - x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right) + \frac{2m}{2m-1} y^{\frac{2m+1}{2m-1}}\left(\frac{dz}{dy}\right)$$

$$- \frac{n}{(2m-1)^2} x^{\frac{2}{2m-1}} z = 0.$$

Coroll. 2.

341. Sumatur porro in priori forma $\nu = \mu$ at fiat $\mu - 2m\mu - 2m = -\mu$, seu $m = \frac{\mu}{\mu+1}$, vt prodeat

$$\frac{1}{bby^{2\mu}}\left(\frac{ddz}{dy^2}\right) - \frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right) - \frac{\mu}{bby^{2\mu+1}}\left(\frac{dz}{dy}\right)$$
$$- \frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right) - \frac{n(\mu+1)^2}{aax^{2\mu+2}} z = 0$$

quae integrabilis existit, quoties fuerit

$$n = -\frac{(\mu+(\mu+1)i)((\mu+1)i+1)}{(\mu+1)^2} \text{ seu}$$
$$n = -\left(i + \frac{\mu}{\mu+1}\right)\left(i + \frac{1}{\mu+1}\right).$$

Scholion.

342. Largissima ergo hinc nobis suppeditatur copia aequationum satis concinnarum, quas ope methodi hic traditae integrare licet. Atque hic imprimis duo casus conspiciuntur, quorum alter

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{4m}{2m-1}} \left(\frac{ddz}{dx^2}\right)$$

pro motu cordarum inaequali crassitie praeditarum determinando est inuentus; alter autem hac aequatione

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right) - \left(\frac{ddz}{dx^2}\right) - \frac{2m}{x}\left(\frac{dz}{dx}\right) = 0$$

contentus ideo est memorabilis, quod in analysi pro soni propagatione instituta, ad talem formam per-

peruenitur. Hae igitur binae aequationes prae ceteris merentur, vt pro casibus integrabilitatis integralia exhibeamus.

Problema 54.

343. Proposita aequatione differentiali

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right)-\left(\frac{ddz}{dx^2}\right)-\frac{2m}{x}\left(\frac{dz}{dx}\right)=0$$

casibus quibus m est numerus integer siue positiuus siue negatiuus, eius integrale completum exhibere.

Solutio.

Facta substitutione $t=\frac{1}{2}x+\frac{b}{2a}y$ et $u=\frac{1}{2}x-\frac{b}{2a}y$, aequatio nostra hanc induit formam

$$\left(\frac{ddz}{dt\,du}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)=0.$$

Cum igitur sit $t+u=x$, si ponamus:

$$\mathfrak{A}=f:\frac{ax+by}{2a}+F:\frac{ax-by}{2a}$$

$$\mathfrak{B}=x\left(f':\frac{ax+by}{2a}+F':\frac{ax-by}{2a}\right)$$

$$\mathfrak{C}=x^2\left(f'':\frac{ax+by}{2a}+F'':\frac{ax-by}{2a}\right)$$

$$\mathfrak{D}=x^3\left(f''':\frac{ax+by}{2a}+F''':\frac{ax-by}{2a}\right)$$

$$\mathfrak{E}=x^4\left(f^{\mathrm{IV}}:\frac{ax+by}{2a}+F^{\mathrm{IV}}:\frac{ax-by}{2a}\right)$$

$$\mathfrak{F}=x^5\left(f^{\mathrm{V}}:\frac{ax+by}{2a}+F^{\mathrm{V}}:\frac{ax-by}{2a}\right)$$

etc.

Casus

Casus integrabiles ita se habebunt, primo negatiui

si $m=0$; $z=\mathfrak{A}$

si $m=-1$; $z=\mathfrak{A}-\frac{1}{2}\mathfrak{B}$

si $m=-2$; $z=\mathfrak{A}-\frac{2}{4}\mathfrak{B}+\frac{1}{4\cdot 3}\mathfrak{C}$

si $m=-3$; $z=\mathfrak{A}-\frac{3}{6}\mathfrak{B}+\frac{3}{6\cdot 5}\mathfrak{C}-\frac{1}{6\cdot 5\cdot 4}\mathfrak{D}$

si $m=-4$; $z=\mathfrak{A}-\frac{4}{8}\mathfrak{B}+\frac{6}{8\cdot 7}\mathfrak{C}-\frac{4}{8\cdot 7\cdot 6}\mathfrak{D}+\frac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E}$

si $m=-5$; $z=\mathfrak{A}-\frac{5}{10}\mathfrak{B}+\frac{10}{10\cdot 9}\mathfrak{C}-\frac{10}{10\cdot 9\cdot 8}\mathfrak{D}+\frac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E}$
$-\frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F}$

etc.

Tum vero pro valoribus positiuis ipsius m

si $m=1$; $xz=\mathfrak{A}$

si $m=2$; $x^3z=\mathfrak{A}-\frac{1}{2}\mathfrak{B}$

si $m=3$; $x^5z=\mathfrak{A}-\frac{2}{4}\mathfrak{B}+\frac{1}{4\cdot 3}\mathfrak{C}$

si $m=4$; $x^7z=\mathfrak{A}-\frac{3}{6}\mathfrak{B}+\frac{3}{6\cdot 5}\mathfrak{C}-\frac{1}{6\cdot 5\cdot 4}\mathfrak{D}$

si $m=5$; $x^9z=\mathfrak{A}-\frac{4}{8}\mathfrak{B}+\frac{6}{8\cdot 7}\mathfrak{C}-\frac{4}{8\cdot 7\cdot 6}\mathfrak{D}+\frac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E}$

si $m=6$; $x^{11}z=\mathfrak{A}-\frac{5}{10}\mathfrak{B}+\frac{10}{10\cdot 9}\mathfrak{C}-\frac{10}{10\cdot 9\cdot 8}\mathfrak{D}+\frac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E}$
$-\frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F}$

etc.

Cui ergo expressioni casu $m=-i$ aequatur valor z, eidem aequatur casu $m=i+1$ valor ipsius $x^{2i+1}z$.

 Scholion.

Scholion.

344. Valores ipsarum t et u ita hic assumsi, vt fieret $t+u=x$; atque eosdem valores quoque in functionibus adhiberi oportet. Etsi enim $f:\frac{ax+by}{2a}$ etiam est functio ipsius $ax+by$, tamen functiones per differentiationem inde deriuatae discrepant. Namque si ponamus

$$f:\frac{ax+by}{2a}=\Phi:(ax+by)$$

erit differentiando

$$\frac{(adx+bdy)}{2a}f':(\frac{ax+by}{2a})=(adx+bdy)\Phi'(ax+by)$$

vnde erit

$$f':\frac{ax+by}{2a}=2a\Phi':(ax+by),$$

neque ergo hae functiones differentiales sunt aequales, etiamsi principales assumtae sint aequales, simili modo erit

$$f'':\frac{(ax+by)}{2a}=4aa\Phi'':(ax+by), \text{ et}$$

$$f''':\frac{ax+by}{2a}=8a^3\Phi''':(ax+by) \text{ etc.}$$

et ita porro.

Problema 55.

345. Proposita aequatione differentiali:

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)$$

casibus quibus m est numerus integer siue positiuus siue negatiuus, integrale completum exhibere.

Solutio.

Solutio.

Introductis nouis variabilibus t et u, ita vt fit

$$t = \tfrac{1}{2} x^{\frac{-1}{2m-1}} - \frac{b}{2(2m-1)a} y \text{ et } u = \tfrac{1}{2} x^{\frac{-1}{2m-1}} + \frac{b}{2(2m-1)a} y$$

aequatio noftra hanc induit formam

$$\left(\frac{ddz}{dt\,du}\right) + \frac{m}{t+u}\left(\frac{dz}{dt}\right) + \frac{m}{t+u}\left(\frac{dz}{du}\right) = 0$$

vbi eft

$$t + u = x^{\frac{-1}{2m-1}}.$$

Pofito igitur:

$$\mathfrak{A} = f:t + \mathrm{F}:u\,;\ \mathfrak{B} = x^{\frac{-1}{2m-1}}(f':t + \mathrm{F}':u)$$

$$\mathfrak{C} = x^{\frac{-2}{2m-1}}(f'':t + \mathrm{F}'':u)\,;\ \mathfrak{D} = x^{\frac{-3}{2m-1}}(f''':t + \mathrm{F}''':u)$$

$$\mathfrak{E} = x^{\frac{-4}{2m-1}}(f^{\mathrm{IV}}:t + \mathrm{F}^{\mathrm{IV}}:u)\,;\ \mathfrak{F} = x^{\frac{-5}{2m-1}}(f^{\mathrm{V}}:t + \mathrm{F}^{\mathrm{V}}:u)$$

etc.

percurramus primo cafus, quibus m a cyphra per numeros negatiuos decrefcit.

I. Si $m = 0$; erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}\left(\frac{ddz}{dx^2}\right) \text{ integrale}$$

$$z = f:\left(\tfrac{1}{2}x + \frac{b}{2a}y\right) + \mathrm{F}:\left(\tfrac{1}{2}x - \frac{b}{2a}y\right).$$

II. Si $m = -1$; ob

$$t = \tfrac{1}{2}x^{\frac{1}{3}} + \frac{b}{6a}y \text{ et } u = \tfrac{1}{2}x^{\frac{1}{3}} - \frac{b}{6a}y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{4}{3}}\left(\frac{ddz}{dx^2}\right) \text{ integrale}$$

$$z = f:t + \mathrm{F}:u - \tfrac{1}{2}x^{\frac{1}{3}}(f':t + \mathrm{F}':u).$$

 III.

III. Si $m = -2$; ob

$$t = \tfrac{1}{2} x^{\frac{1}{5}} + \frac{b}{10a} y \quad \text{et} \quad u = \tfrac{1}{2} x^{\frac{1}{5}} - \frac{b}{10a} y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{8}{5}} \left(\frac{ddz}{dx^2}\right) \text{ integrale}$$

$$z = f:t + F:u - \tfrac{2}{4} x^{\frac{1}{5}} (f':t + F':u) + \tfrac{1}{4 \cdot 3} x^{\frac{2}{5}} (f'':t + F'':u).$$

IV. Si $m = -3$; ob

$$t = \tfrac{1}{2} x^{\frac{1}{7}} + \frac{b}{14a} y \quad \text{et} \quad u = \tfrac{1}{2} x^{\frac{1}{7}} - \frac{b}{14a} y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{12}{7}} \left(\frac{ddz}{dx^2}\right) \text{ integrale}$$

$$z = f:t + F:u - \tfrac{3}{6} x^{\frac{1}{7}} (f':t + F':u) + \tfrac{3}{6 \cdot 5} x^{\frac{2}{7}} (f'':t + F'':u)$$
$$- \tfrac{1}{6 \cdot 5 \cdot 4} x^{\frac{3}{7}} (f''':t + F''':u).$$

V. Si $m = -4$; ob

$$t = \tfrac{1}{2} x^{\frac{1}{9}} + \frac{b}{18a} y \quad \text{et} \quad u = \tfrac{1}{2} x^{\frac{1}{9}} - \frac{b}{18a} y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{16}{9}} \left(\frac{ddz}{dx^2}\right) \text{ integrale}$$

$$z = f:t + F:u - \tfrac{4}{8} x^{\frac{1}{9}} (f':t + F':u) + \tfrac{6}{8 \cdot 7} x^{\frac{2}{9}} (f'':t + F'':u)$$
$$- \tfrac{4}{8 \cdot 7 \cdot 6} x^{\frac{3}{9}} (f''':t + F''':u) + \tfrac{1}{8 \cdot 7 \cdot 6 \cdot 5} x^{\frac{4}{9}} (f^{\mathrm{IV}}:t + F^{\mathrm{IV}}:u)$$

et ita porro.

Pro

Pro altero vero casu vbi m habet valores positiuos, integralia sequenti modo exprimentur:

I. Si sit $m = 1$, seu $\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^4 \left(\frac{ddz}{dx^2}\right)$

ob $t = \frac{1}{2} x^{-1} - \frac{b}{2a} y$ et $u = \frac{1}{2} x^{-1} + \frac{b}{2a} y$

erit integrale

$$x^{-1} z = f:t + F:u \text{ seu } z = x(f:t + F:u).$$

II. Si sit $m = 2$, seu $\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{8}{3}} \left(\frac{ddz}{dx^2}\right)$

ob $t = \frac{1}{2} x^{-\frac{1}{3}} - \frac{b}{6a} y$ et $u = \frac{1}{2} x^{-\frac{1}{3}} + \frac{b}{6a} y$

erit integrale

$$z = x(f:t + F:u) - \frac{1}{2} x^{\frac{2}{3}} (f':t + F':u).$$

III. Si sit $m = 3$, seu $\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{12}{5}} \left(\frac{ddz}{dx^2}\right)$

ob $t = \frac{1}{2} x^{-\frac{1}{5}} - \frac{b}{10a} y$ et $u = \frac{1}{2} x^{-\frac{1}{5}} + \frac{b}{10a} y$

erit integrale

$$z = x(f:t + F:u) - \frac{2}{4} x^{\frac{4}{5}} (f':t + F':u) + \frac{1}{4 \cdot 8} x^{\frac{3}{5}} (f'':t + F'':u).$$

IV. Si sit $m = 4$, seu $\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{16}{7}} \left(\frac{ddz}{dx^2}\right)$

ob $t = \frac{1}{2} x^{-\frac{1}{7}} - \frac{b}{14a} y$ et $u = \frac{1}{2} x^{-\frac{1}{7}} + \frac{b}{14a} y$

erit integrale

$$z = x(f:t + F:u) - \frac{3}{6} x^{\frac{6}{7}} (f':t + F':u) + \frac{3}{6 \cdot 5} x^{\frac{5}{7}} (f'':t + F'':u)$$
$$- \frac{1}{6 \cdot 5 \cdot 4} x^{\frac{4}{7}} (f''':t + F''':u).$$

V. Si fit $m=5$, feu $\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{20}{9}}\left(\frac{ddz}{dx^2}\right)$

ob $t=\frac{1}{2}x^{-\frac{1}{9}}-\frac{b}{18a}y$ et $u=\frac{1}{2}x^{-\frac{1}{9}}+\frac{b}{18a}y$

erit integrale

$$z=x(f:t+F:u)-\tfrac{4}{8}x^{\frac{8}{9}}(f':t+F':u)+\tfrac{6}{8\cdot 7}x^{\frac{7}{9}}(f'':t+F'':u)$$
$$-\tfrac{4}{8\cdot 7\cdot 6}x^{\frac{6}{9}}(f''':t+F''':u)+\tfrac{1}{8\cdot 7\cdot 6\cdot 5}x^{\frac{5}{9}}(f^{IV}:t+F^{IV}u).$$

etc.

vnde lex, qua has expreffiones vlterius continuare licet, per fe eft manifefta.

Scholion 1.

346. Cafus ifti integrabilitatis congruunt cum iis, qui in aequatione *Riccatiana* dicta deprehenduntur, nouimus fcilicet aequationem hanc

$$dy+yydx=ax^{\frac{-4m}{2m-1}}dx$$

integrari poffe quoties m eft numerus integer fiue pofitiuus fiue negatiuus. Haec autem aequatio haud leui vinculo cum noftra forma eft connexa, quod ita oftendi poteft. Propofita forma generali

$$\left(\frac{ddz}{dy^2}\right)=X\left(\frac{ddz}{dx^2}\right)$$

pro integralibus particularibus inueniendis ftatuatur $z=e^{\alpha y}v$, vt v fit functio ipfius x tantum, erit

$$\left(\frac{dz}{dx}\right)=e^{\alpha y}.\frac{dv}{dx} \text{ et } \left(\frac{ddz}{dx^2}\right)=e^{\alpha y}.\frac{ddv}{dx^2};$$

tum

tum vero $(\frac{ddz}{dy^2}) = \alpha\alpha e^{\alpha y} v$ vnde prodit haec aequatio $\alpha\alpha v = \frac{X ddv}{dx^2}$; in qua ſi porro ſtatuatur $v = e^{\int p dx}$ oritur $\frac{\alpha\alpha dx}{X} = dp + ppdx$; ac ſi $X = Ax^{\frac{4m}{2m-1}}$ vt in noſtro caſu haec aequatio fit

$$dp + ppdx = ax^{\frac{-4m}{2m-1}} dx.$$

Haud temere igitur euenire putandum eſt, quod vtraque aequatio iisdem caſibus integrationem admittat. Interim tamen notatu dignum occurrit, quod caſus $m = \infty$, qui in forma *Riccatiana* fit facillimus, idem in noſtra aequatione neutiquam integrationem admittat. Habetur quippe haec aequatio

$$(\frac{ddz}{dy^2}) = \frac{bb}{aa} xx (\frac{ddz}{dx^2}),$$

cuius reductio modo ſupra §. 330. adhibito non ſuccedit. Nam ob

$$Q = \frac{bx}{a},\ R = 0,\ S = 0 \text{ et } T = 0,$$

pro nouis variabilibus ponitur

$$t = \int p(dx + \frac{bxdy}{a}) \text{ et } u = \int q(dx - \frac{bxdy}{a});$$

vnde ob $M = \frac{bbx}{aa} = N$ oritur haec aequatio

$$(\frac{ddz}{dt\,du}) - \frac{1}{4qx}(\frac{dz}{dt}) - \frac{1}{4px}(\frac{dz}{du}) = 0$$

quae ſumendo

$$p = \frac{1}{x} \text{ et } q = \frac{1}{x}$$

vt fit

$$t = lx + \frac{by}{a} \text{ et } u = lx - \frac{by}{a},$$

tranſit

transit in

$$\left(\frac{ddz}{dt\,du}\right)-\tfrac{1}{4}\left(\frac{dz}{dt}\right)-\tfrac{1}{4}\left(\frac{dz}{du}\right)=0$$

cuius integratio haud perspicitur.

Scholion 2.

347. Aequationis autem $\left(\frac{ddz}{dy^2}\right)=xx\left(\frac{ddz}{dx^2}\right)$ integralia particularia infinita exhibere licet, in hac forma $z=Ax^\lambda e^{\mu y}$ contenta. Cum enim hinc fit:

$$\left(\frac{dz}{dy}\right)=\mu A x^\lambda e^{\mu y} \text{ et } \left(\frac{dz}{dx}\right)=\lambda A x^{\lambda-1} e^{\mu y} \text{ erit}$$

$$\mu\mu A x^\lambda e^{\mu y}=\lambda(\lambda-1)A x^\lambda e^{\mu y} \text{ ideoque}$$

$\mu=\sqrt{\lambda(\lambda-1)}$, vnde ex quouis numero pro λ assumto bini valores pro μ oriuntur ita vt habeatur

$$z=Ax^\lambda e^{y\sqrt{\lambda(\lambda-1)}}+Bx^\lambda e^{-y\sqrt{\lambda(\lambda-1)}},$$

et huiusmodi membrorum numerus variando λ in infinitum multiplicari potest. Interim tamen singula haec membra adhuc generaliora reddi possunt. Posito enim $z=x^\lambda e^{\mu y}v$, videamus an v necessario constans esse debeat: hinc autem fit

$$\left(\frac{dz}{dy}\right)=\mu x^\lambda e^{\mu y}v+x^\lambda e^{\mu y}\left(\frac{dv}{dy}\right) \text{ et}$$
$$\left(\frac{dz}{dx}\right)=\lambda x^{\lambda-1}e^{\mu y}v+x^\lambda e^{\mu y}\left(\frac{dv}{dx}\right)$$

ideoque nostra aequatio praebet per $x^\lambda e^{\mu y}$ diuisa

$$\mu\mu v+2\mu\left(\frac{dv}{dy}\right)+\left(\frac{ddv}{dy^2}\right)=\lambda(\lambda-1)v+2\lambda x\left(\frac{dv}{dx}\right)+xx\left(\frac{ddv}{dx^2}\right).$$

Statuatur vt ante $\mu\mu=\lambda(\lambda-1)$, sitque $v=\alpha lx+\beta y$, erit

$$2\beta\mu=2\alpha\lambda-\alpha \text{ seu } \frac{\alpha}{\beta}=\frac{2\mu}{2\lambda-1}=\frac{2\sqrt{\lambda(\lambda-1)}}{2\lambda-1}$$

vnde

vnde cuiusque membri ex numero λ nati forma erit:

$$z = x^{\lambda}\left(e^{y\sqrt{\lambda(\lambda-1)}}\left(\mathrm{A} + \frac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{A}}lx + \frac{2\lambda-1}{\mathfrak{A}}y\right) + e^{-y\sqrt{\lambda(\lambda-1)}}\left(\ldots\ \mathfrak{B}\ldots lx + \frac{2\lambda-1}{\mathfrak{B}}y\right)\right).$$

Quomodocunque igitur non ſolum exponens λ ſed etiam quantitates A, $\mathfrak{A}$, B, $\mathfrak{B}$ varientur, infinita huiusmodi membra formari poſſunt, quae omnia iunctim ſumta valorem completum functionis z praebere ſunt cenſenda. Quin etiam pro λ imaginaria aſſumi poſſunt, poſito enim

$$\lambda = a + b\sqrt{-1} \text{ fit } \mu = p + q\sqrt{-1}$$

exiſtente

$$pp - qq = aa - a - bb \text{ et}$$
$$pp + qq = \sqrt{(aa+bb)(aa-2a+1+bb)}$$

tum vero eſt

$$x^{\lambda} = x^{a}(\operatorname{cof} blx + \sqrt{-1}.\operatorname{fin}.blx) \text{ et}$$
$$e^{\mu y} = e^{py}(\operatorname{cof}.qy + \sqrt{-1}.\operatorname{fin}.qy),$$

vnde colligitur forma realis:

$$z = x^{a}e^{py}\begin{pmatrix} \mathrm{A}\operatorname{cof}.(blx+qy) + \mathrm{B}(2plx+(2a-1)y)\operatorname{cof}.(blx+qy) - \mathrm{B}'2qlx+2by)\operatorname{fin}.(blx+qy) \\ \mathfrak{A}\operatorname{fin}.(blx+qy) + \mathfrak{B}(2plx+(2a-1)y)\operatorname{fin}.(blx+qy) + \mathfrak{B}(2qlx+2by)\operatorname{cof}.(blx+qy) \end{pmatrix}$$

vbi quantitates a et b pro lubitu aſſumere licet, vnde ſimul p et q definiuntur. Quodſi hic litteras b et q vt datas ſpectemus, binae reliquae a et p ex iis ita determinantur, vt ſit

$$2a - 1 = q\sqrt{\left(\frac{1}{qq-bb} - 4\right)} \text{ et } p = \frac{b}{2}\sqrt{\left(\frac{1}{qq-bb} - 4\right)}$$

hic ergo necesse est sit $qq > bb$ et $qq < bb + \frac{1}{4}$, seu qq inter hos arctos limites bb et $bb + \frac{1}{4}$ contineri debet statuatur $q = c$ et $\sqrt{\left(\frac{1}{qq - bb} - 4\right)} = 2f$, vt sit

$$\frac{1}{qq - bb} = 4(1 + ff) \text{ seu } \ldots$$

atque $2a - 1 = 2cf$ et $p = bf$

ex quo forma integralium particularium erit

$$z = x^{cf + \frac{1}{2}} e^{bfy} \left(\begin{array}{l} \mathrm{A} \operatorname{cos.}(blx + cy) + 2\mathrm{B}f(blx + cy) \operatorname{cos.}(blx + cy) - 2\mathrm{B}(clx + by) \operatorname{sin.} blx + cy \\ \mathfrak{A} \operatorname{sin.}(blx + cy) + 2\mathfrak{B}f(blx + cy) \operatorname{sin.}(blx + cy) + 2\mathfrak{B}(clx + by) \operatorname{cos.} blx + cy \end{array} \right)$$

quae posito breuitatis gratia angulo $blx + cy = \Phi$ transformatur in hanc

$$z = x^{cf + \frac{1}{2}} e^{bfy} \left(\mathrm{A} \operatorname{cos.}(\Phi + \alpha) + \mathrm{B}f(blx + cy) \operatorname{sin.}(\Phi + \beta) + \mathrm{B}(clx + by) \operatorname{cos.}(\Phi + \beta) \right)$$

vbi quantitates b, c, A, B, α, β ab arbitrio nostro pendent.

Scholion 3.

348. Resolutio ergo aequationis

$$\left(\frac{ddz}{dy^2}\right) = xx\left(\frac{ddz}{dx^2}\right)$$

ita institui potest, vt fingatur

$$z = x^{\lambda} e^{\mu y} (mlx + ny),$$

vnde fit

$$\left(\frac{dz}{dx}\right) = \lambda x^{\lambda - 1} e^{\mu y} (mlx + ny) + m x^{\lambda - 1} e^{\mu y} \text{ et}$$

$$\left(\frac{dz}{dy}\right) = \mu x^{\lambda} e^{\mu y} (mlx + ny) + n x^{\lambda} e^{\mu y}$$

hinc-

hincque vlterius differentiando:

$$\left(\frac{ddz}{dx^2}\right) = x^{\lambda-2} e^{\mu y}(m(2\lambda-1)+\lambda(\lambda-1)mlx+\lambda(\lambda-1)ny) \text{ et}$$
$$\left(\frac{ddz}{dy^2}\right) = x^{\lambda} e^{\mu y}(2\mu n+\mu\mu mlx+\mu\mu ny).$$

Ex quo colligitur primo $\mu = \sqrt{\lambda(\lambda-1)}$, deinde $2n\sqrt{\lambda(\lambda-1)} = m(2\lambda-1)$ vt sit $\frac{m}{n} = \frac{2\sqrt{\lambda(\lambda-1)}}{2\lambda-1}$, sicque eadem prodit integratio quam modo ante dedimus.

CAPVT V.

TRANSFORMATIO SINGVLARIS EARVNDEM AEQVATIONVM.

Problema 56.

349.

Propofita hac aequatione

$$\left(\frac{ddz}{dy^2}\right)=P\left(\frac{ddz}{dx^2}\right)+Q\left(\frac{dz}{dx}\right)+Rz$$

in qua P, Q, R fint functiones ipfius x tantum, eam ope fubftitutionis

$$z=M\left(\frac{dv}{dx}\right)+Nv,$$

vbi quoque fint M et N functiones ipfius x tantum, in aliam eiusdem formae transmutare vt prodeat:

$$\left(\frac{ddv}{dy^2}\right)=F\left(\frac{ddv}{dx^2}\right)+G\left(\frac{dv}{dx}\right)+Hv$$

exiftentibus F, G, H functionibus folius x.

Solutio.

Quia quantitates M et N ab y funt immunes erit

$$\left(\frac{ddz}{dy^2}\right)=M\left(\frac{d^3v}{dxdy^2}\right)+N\left(\frac{ddv}{dy^2}\right)$$

quae

quae forma per aequationem, quam tandem resultare assumimus, abit in hanc:

$$\left(\frac{ddz}{dy^2}\right)=MF\left(\frac{d^3v}{dx^3}\right)+\frac{MdF}{dx}\left(\frac{ddv}{dx^2}\right)+\frac{MdG}{dx}\left(\frac{dv}{dx}\right)+\frac{MdH}{dx}v$$
$$+MG \quad +MH \quad +NH$$
$$+NF \quad +NG.$$

Deinde vero pro altero aequationis propositae membro nostra substitutio praebet:

$$\left(\frac{dz}{dx}\right)=M\left(\frac{ddv}{dx^2}\right)+\frac{dM}{dx}\left(\frac{dv}{dx}\right)+\frac{dN}{dx}v$$
$$+N$$

hincque porro

$$\left(\frac{ddz}{dx^2}\right)=M\left(\frac{d^3v}{dx^3}\right)+\left(\frac{2dM}{dx}+N\right)\left(\frac{ddv}{dx^2}\right)$$
$$+\left(\frac{ddM}{dx^2}+\frac{2dN}{dx^2}\right)\left(\frac{dv}{dx}\right)+\frac{ddN}{dx^2}v.$$

Cum nunc sit per hypothesin:

$$\left(\frac{ddz}{dy^2}\right)=P\left(\frac{ddz}{dx^2}\right)+Q\left(\frac{dz}{dx}\right)+Rz$$

si hic valores modo inventi substituantur, singulaque membra $\left(\frac{d^3v}{dx^3}\right)$; $\left(\frac{ddv}{dx^2}\right)$; $\left(\frac{dv}{dx}\right)$ et v seorsim ad nihilum redigantur, quatuor sequentes aequationes orientur, scilicet

$$\begin{array}{c|l}
\text{ex} & \text{colligitur aequatio} \\
\left(\frac{d^3v}{dx^3}\right) & MF=MP \\
\left(\frac{ddv}{dx^2}\right) & \frac{MdF}{dx}+MG+NF=\left(\frac{2dM}{dx}+N\right)P+MQ \\
\left(\frac{dv}{dx}\right) & \frac{MdG}{dx}+MH+NG=\left(\frac{ddM}{dx^2}+\frac{2dN}{dx}\right)P+\left(\frac{dM}{dx}+N\right)Q+MR \\
v & \frac{MdH}{dx}+NH=\frac{ddN}{dx^2}P+\frac{dN}{dx}Q+NR
\end{array}$$

ex quibus commodissime primo quaeruntur P, Q et R. Verum prima dat statim $P = F$, vnde secunda fit

$$\frac{MdF - 2FdM}{Mdx} + G = Q.$$

Ex binis vltimis autem eliminando R colligitur:

$$\frac{M(NdG - MdH)}{dx} + NNG = \left(\frac{NddM - MddN}{dx^2} + \frac{2NdN}{dx}\right)F$$
$$+ \left(\frac{NdM - MdN}{dx} + NN\right)Q$$

et illum valorem pro Q substituendo:

$$0 = \frac{MMdH}{dx} - \frac{MNdG}{dx} + \frac{(NddM - MddN)}{dx^2}F + \frac{2NFdN}{dx}$$
$$+ \frac{NdM - MdN}{dx}G + \frac{(NdM - MdN)}{dx^2}dF + \frac{NNdF}{dx}$$
$$- \frac{2FdM(NdM - MdN)}{Mdx^2} - \frac{2NNFdM}{Mdx}$$

quae aequatio per $\frac{dx}{MM}$ multiplicata commode integrabilis redditur, inueniturque integrale:

$$C = H - \frac{N}{M}G + \frac{NdM - MdN}{MMdx}F + \frac{NNF}{MM}.$$

Quod si ergo breuitatis gratia ponamus $N = Ms$ erit

$$C = H - Gs - F\frac{ds}{dx} + Fss \text{ seu}$$

$$ds + \frac{G}{F}sdx - ssdx + \frac{(C - H)dx}{F} = 0.$$

Siue iam hinc definiatur quantitas $s = \frac{N}{M}$ siue vna functionum F, G et H, pro ipsa aequatione proposita litterae P, Q et R, ita determinabuntur, vt sit

I. $P = F$

II. $Q = G + \frac{dF}{dx} - \frac{2FdM}{Mdx}$

et

et ex vltima aequatione deriuatur

$$R = H + \frac{MdH}{Ndx} - \frac{FddN}{Ndx^2} - \frac{dN}{Ndx}\left(G + \frac{dF}{dx} - \frac{2FdM}{Mdx}\right)$$

qui valor ob $N = Ms$ euadit:

$$R = H + \frac{dH}{sdx} - \frac{Gds}{sdx} - \frac{GdM}{Mdx} - \frac{Fdds}{sdx^2} - \frac{FddM}{Mdx^2} + \frac{2FdM^2}{MMdx^2}$$

et cum aequatio inuenta, si differentietur det

$$0 = dH - Gds - sdG - \frac{Fdds}{dx} - \frac{dFds}{dx} + 2Fsds + ssdF$$

obtinebimus

$$\text{III. } R = H - \frac{GdM}{Mdx} + \frac{dG}{dx} - \frac{FddM}{Mdx^2} - \frac{2Fds}{dx} + \frac{2FdM^2}{MMdx^2} - \frac{sdF}{dx} - \frac{dFdM}{Mdx^2}$$

vnde si aequatio

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

resolutionem admittat, etiam resolutio succedet huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz$$

cum sit

$$z = M\left(\frac{dv}{dx}\right) + Nv = M\left(sv + \left(\frac{dv}{dx}\right)\right).$$

Coroll. 1.

350. Si ponatur $M = 1$ vt fiat $z = sv + \left(\frac{dv}{dx}\right)$ erit

$$P = F, \quad Q = G + \frac{dF}{dx}; \text{ et } R = H + \frac{dG}{dx} - \frac{2Fds - sdF}{dx}$$

neque

neque hoc modo vsus istius reductionis restringitur; quoniam si deinceps loco z ponatur Mz, etiam aequationis hinc ortae resolutio est in promtu.

Corollarium

$$\left(\frac{ddv}{dy^2}\right)=F\left(\frac{ddv}{dx^2}\right)+G\left(\frac{dv}{dx}\right)+Hv$$

resolutio est in potestate, toties etiam huius aequationis:

$$\left(\frac{ddz}{dy^2}\right)=F\left(\frac{ddz}{dx^2}\right)+\left(G+\frac{dF}{dx}\right)\left(\frac{dz}{dx}\right)+\left(H+\frac{dG}{dx}-\frac{2Fds-sdF}{dx}\right)z$$

resolutio succedit, si modo capiatur s ex hac aequatione

$$Fds+Gsdx-Fssdx+(C-H)dx=0$$

tum enim erit $z=sv+\left(\frac{dv}{dx}\right)$. Sunt autem litterae F, G, H functiones ipsius x tantum.

Scholion.

352. Haec reductio methodum maxime naturalem suppeditare videtur eiusmodi integrationes perficiendi, quae simul functionum differentialia inuoluunt. Si enim aequationis pro v datae integrale sit $v=\Phi:t$ existente t functione ipsarum x et y, ob $dv=dt\Phi':t$ erit $\left(\frac{dv}{dx}\right)=\left(\frac{dt}{dx}\right)\Phi':t$ aequationis inde deriuatae pro z habebimus:

$$z=s\Phi:t+\left(\frac{dt}{dx}\right)\Phi':t.$$

Deinde

Deinde si fuerit generalius $v = u\Phi : t$ fiet:

$$z = s u \Phi : t + \left(\frac{du}{dx}\right)\Phi : t + u\left(\frac{dt}{dx}\right)\Phi' : t$$

vnde ratio perspicitur ad eiusmodi aequationes perveniendi, quarum integralia praeter functionem $\Phi : t$ etiam functiones ex eius differentiatione natas $\Phi' : t$, atque adeo etiam sequentes $\Phi'' : t$, $\Phi''' : t$ etc. complectantur. Quamobrem operae pretium erit hanc reductionem accuratius euoluere.

Problema 57.

353. Concessa resolutione huius aequationis:

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{m}{x}\left(\frac{dv}{dx}\right) + \frac{n}{xx}v$$

inuenire aliam aequationem huius formae

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz$$

pro qua sit

$$z = sv + \left(\frac{dv}{dx}\right).$$

Solutio.

Facta comparatione cum praecedente problemate habemus:

$$F = 1, \; G = \frac{m}{x} \text{ et } H = \frac{n}{xx}$$

vnde quantitatem s ex hac aequatione definiri oportet

$$ds + \frac{msdx}{x} - ssdx + \left(f - \frac{n}{xx}\right)dx = 0$$

qua inuenta ob $\frac{dG}{dx} = -\frac{m}{xx}$, aequatio quaesita erit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \left(\frac{n-m}{xx} - \frac{2ds}{dx}\right)z$$

seu loco ds valore inde substituto:

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \left(2f - \frac{m-n}{xx} + \frac{2ms}{x} - 2ss\right)z$$

pro qua est

$$z = sv + \left(\frac{dv}{dx}\right).$$

I. Ponamus primo quantitatem constantem $f = 0$, vt sit

$$ds + \frac{msdx}{x} - ssdx - \frac{ndx}{xx} = 0$$

cuius integrale particulare est $s = \frac{\alpha}{x}$ existente:

$$-\alpha + m\alpha - \alpha\alpha - n = 0, \text{ seu } \alpha\alpha - (m-1)\alpha + n = 0$$

ex quo ob $\frac{ds}{dx} = \frac{-\alpha}{xx}$ oritur haec aequatio

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \frac{2\alpha - m + n}{xx}z$$

pro qua est

$$z = \frac{\alpha}{x}v + \left(\frac{dv}{dx}\right)$$

seu exclusa $n = \alpha(m-1-\alpha)$, si constet resolutio huius aequationis:

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{m}{x}\left(\frac{dv}{dx}\right) + \frac{\alpha(m-1-\alpha)}{xx}v$$

pro hac

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \frac{(\alpha-1)(m-\alpha)}{xx}z$$

erit

$$z = \frac{\alpha}{x}v + \left(\frac{dv}{dx}\right).$$

II.

II. Maneat $f=0$, et quaeramus pro s valorem completum ponendo $s=\frac{\alpha}{x}+\frac{1}{t}$, fietque ob

$$n=(m-1)\alpha-\alpha\alpha;\; dt+\frac{(2\alpha-m)t\,dx}{x}+dx=0$$

quae per $x^{2\alpha-m}$ multiplicata et integrata praebet:

$$t=\frac{cx^{m-2\alpha}}{2\alpha-m+1}-\frac{x}{2\alpha-m+1}$$

hincque

$$s=\frac{\alpha cx^{m-2\alpha-1}+\alpha-m+1}{x(cx^{m-2\alpha-1}-1)}=\frac{\alpha}{x}+\frac{2\alpha-m+1}{x(cx^{m-2\alpha-1}-1)}$$

vnde fit

$$\frac{ds}{dx}=\frac{-\alpha}{xx}+\frac{(m-2\alpha-1)(m-2\alpha)}{xx(cx^{m-2\alpha-1}-1)}+\frac{(m-2\alpha-1)^2}{xx(cx^{m-2\alpha-1}-1)^2}.$$

Hic praecipue notetur casus $c=0$ quo fit

$$s=\frac{m-\alpha-1}{x} \text{ et } \frac{ds}{dx}=\frac{-m+\alpha+1}{xx},$$

ita vt data aequatione

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{m}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(m-1-\alpha)}{xx}v$$

pro hac aequatione

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha+1)(m-2-\alpha)}{xx}z$$

futurum fit

$$z=\frac{m-\alpha-1}{x}v+\left(\frac{dv}{dx}\right).$$

Pro generali autem valore fit $m-2\alpha-1=\beta$, vt habeatur

$$s=\frac{\alpha}{x}-\frac{\beta}{x(cx^{\beta}-1)} \text{ et } \frac{ds}{dx}=\frac{-\alpha}{xx}+\frac{\beta(\beta+1)}{xx(cx^{\beta}-1)}$$
$$+\frac{\beta\beta}{xx(cx^{\beta}-1)^2}$$

vnde si detur haec aequatio

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{2\alpha+\beta+1}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(\alpha+\beta)}{xx}v$$

eius ope resoluetur haec:

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\beta+1}{x}\left(\frac{dz}{dx}\right)+\Big((\alpha-1)(\alpha+\beta+1)-\frac{2\beta(\beta+1)}{cx^{\beta}-1}$$
$$-\frac{2\beta\beta}{(cx^{\beta}-1)^2}\Big)\frac{z}{xx}$$

cum sit

$$z=\left(\alpha-\frac{\beta}{cx^{\beta}-1}\right)\frac{v}{x}+\left(\frac{dv}{dx}\right).$$

III. Rationem quoque habeamus constantis f, ponamusque $f=\frac{1}{aa}$, vt facto $n=\alpha(m-1-\alpha)$ habeamus

$$ds+\frac{msdx}{x}-ssdx-\frac{\alpha(m-1-\alpha)dx}{xx}+\frac{dx}{aa}=0$$

quae posito $s=\frac{\alpha}{x}+\frac{1}{t}$ abit in

$$dt-\frac{(m-2\alpha)tdx}{x}+dx=\frac{tt}{aa}dx.$$

Sit $m-2\alpha=\gamma$ vt aequatio data sit:

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{2\alpha+\gamma}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(\alpha+\gamma-1)}{xx}v$$

et

et inuenta quantitate s prodeat haec aequatio:

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\gamma}{x}\left(\frac{dz}{dx}\right)+\left(\frac{\alpha\alpha-2\alpha+\alpha\gamma-\gamma}{xx}-\frac{2ds}{dx}\right)z$$

feu

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\gamma}{x}\left(\frac{dz}{dx}\right)+\left(\frac{(\alpha-1)(\alpha+\gamma)}{xx}+\frac{2dt}{tt\,dx}\right)z$$

pro qua eft

$$z=\left(\frac{\alpha}{x}+\frac{1}{t}\right)v+\left(\frac{dv}{dx}\right)$$

vbi totum negotium ad inuentionem quantitatis t redit ex aequatione

$$dt-\frac{\gamma t\,dx}{x}+dx=\frac{tt}{aa}dx.$$

Hunc in finem ftatuatur $t=a-\frac{aadu}{udx}$, ac reperitur:

$$\frac{ddu}{dx^2}-\frac{\gamma du}{xdx}-\frac{2du}{adx}+\frac{\gamma u}{ax}=0$$

cuius duplex refolutio datur altera ponendo

$u=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5$ etc. exiftente

$B=\frac{\gamma A}{\gamma a}$; $C=\frac{(\gamma-2)B}{2(\gamma-1)a}$; $D=\frac{(\gamma-4)C}{3(\gamma-2)a}$; $E=\frac{(\gamma-6)D}{4(\gamma-3)a}$ etc.

altera vero ponendo:

$u=Ax^{\gamma+1}+Bx^{\gamma+2}+Cx^{\gamma+3}+Dx^{\gamma+4}+Ex^{\gamma+5}$ etc. vbi

$$B=\frac{(\gamma+2)A}{(\gamma+2)a};\quad C=\frac{(\gamma+4)B}{2(\gamma+3)a};\quad D=\frac{(\gamma+6)C}{3(\gamma+4)a};$$
$$E=\frac{(\gamma+8)D}{4(\gamma+5)a}\ \text{etc.}$$

quarum illa abrumpitur fi fit γ numerus integer par pofitiuus, haec vero fi negatiuus. Qui valores etfi funt particulares, tamen fupra iam oftendimus quomodo inde valores completi fint eliciendi.

 Coroll. 1.

Coroll. 1.

354. Supra autem vidimus (333) hanc aequationem

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{2m}{x}\left(\frac{dv}{dx}\right)+\frac{(m+i)(m-i-1)}{xx}v$$

esse integrabilem si sit i numerus integer quicunque, vnde colligimus, hanc aequationem:

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{m}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(m-1-\alpha)}{xx}v$$

integrationem admittere quoties fuerit vel $\alpha=\frac{1}{2}m+i$ vel $\alpha=\frac{1}{2}m-i-1$, seu $m-2\alpha$ numerus integer par siue positiuus siue negatiuus, qui casus ob $m-2\alpha=\gamma$ cum casibus integrabilitatis, pro valore generali ipsius s inueniendo congruunt.

Coroll. 2.

355. Quando autem ex hac aequatione functionem v definire licet, tum etiam hae duae sequentes aequationes illi similes resolui poterunt:

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha-1)(m-\alpha)}{xx}z \quad \text{et}$$

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha+1)(m-\alpha-1)}{xx}z$$

cum pro illa sit

$$z=\frac{\alpha}{x}v+\left(\frac{dv}{dx}\right);$$

pro hac vero

$$z=\frac{m-\alpha-1}{x}v+\left(\frac{dv}{dx}\right).$$

Coroll. 3.

Coroll. 3.

356. Praeterea vero etiam aequationes alius generis, vbi postremus terminus non est formae $\frac{n}{xx}z$, resolui possunt qui inueniuntur, si quantitatis s valor generalius inuestigatur, atque adeo constantis f ratio habetur.

Exemplum 1.

357. *Proposita aequatione* $\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)$ *pro qua est*

$$v=\pi:(x+y)+\Phi:(x-y)$$

inuenire aequationes magis complicatas, quae huius ope integrari queant.

Cum hic sit $F=1$, $G=0$ et $H=0$, resoluatur haec aequatio

$$ds-ss\,dx+C\,dx=0$$

et huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{2ds}{dx}z$$

integrale erit

$$z=sv+\left(\frac{dv}{dx}\right).$$

Sumta autem primo constante $C=0$, fit $\frac{ds}{ss}=dx$ et $\frac{1}{s}=c-x$ seu $s=\frac{1}{c-x}$ atque $\frac{ds}{dx}=\frac{1}{(c-x)^2}$, vbi quidem sine vlla restrictione poni potest $c=0$, vt huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{2}{xx}z$$

inte-

integrale fit

$$z = -\tfrac{1}{x}(\pi:(x+y)+\Phi:(x-y)) + \pi':(x+y)+\Phi':(x-y).$$

Sit deinde $C = aa$, et ob $ds = dx(ss-aa)$ fiet

$$x = \tfrac{1}{2a}\, l\, \tfrac{s-a}{s+a}\text{, hincque}$$

$$\frac{s-a}{s+a} = Ae^{2ax} \text{ et } s = \frac{a(1+Ae^{2ax})}{1-Ae^{2ax}} \text{ vnde}$$

$$\frac{ds}{dx} = \frac{4Aaae^{2ax}}{(1-Ae^{2ax})^2},$$

et aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{8Aaae^{2ax}}{(1-Ae^{2ax})^2}z$$

integrale est

$$z = \frac{a(1+Ae^{2ax})}{1-Ae^{2ax}}v + \left(\frac{dv}{dx}\right).$$

Sit tandem $C = -aa$ et ob $ds = dx(aa+ss)$ fit

$$ax+b = \text{Ang. tang.} \tfrac{s}{a},$$

hincque

$$s = a\,\text{tang.}(ax+b) \text{ et } \tfrac{ds}{dx} = \tfrac{aa}{\text{cof.}(ax+b)^2},$$

quocirca huius aequationis:

$$\left(\tfrac{ddz}{dy^2}\right) = \left(\tfrac{ddz}{dx^2}\right) - \tfrac{2aa}{\text{cof.}(ax+b)^2}z$$

integrale est

$$z = \tfrac{a\,\text{fin.}(ax+b)}{\text{cof.}(ax+b)}v + \left(\tfrac{dv}{dx}\right).$$

Exem-

Exemplum 2.

359. *Proposita aequatione*

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)-\frac{2}{xx}v,$$

cuius integrale constat, inuenire alias eius ope integrabiles.

Pro hoc casu habemus:

$$ds-ssdx+\left(C+\frac{2}{xx}\right)dx=0$$

qua resoluta erit huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-2\left(\frac{1}{xx}+\frac{ds}{dx}\right)z$$

integrale

$$z=sv+\left(\frac{dv}{dx}\right).$$

I. Sit primo $C=0$, et ex aequatione

$$ds-ssdx+\frac{2dx}{xx}=0$$

fit particulariter $s=\frac{1}{x}$ vel $s=-\frac{2}{x}$. Ponatur ergo $s=\frac{1}{x}+\frac{1}{t}$ eritque

$$dt+\frac{2tdx}{x}+dx=0,$$

hinc

$$txx+\tfrac{1}{3}x^3=\tfrac{1}{3}a^3.$$

Ergo

$$t=\frac{a^3-x^3}{3xx} \text{ et } s=\frac{a^3+2x^3}{x(a^3-x^3)};$$

ideoque

$$\frac{ds}{dx}+\frac{1}{xx}=\frac{3x(2a^3+x^3)}{(a^3-x^3)^2}$$

 vnde

vnde huius aequationis:

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{6x(2a^3+x^3)}{(a^3-x^3)^2}z$$

integrale est

$$z = \frac{a^3+2x^3}{x(a^3-x^3)}v + \left(\frac{dv}{dx}\right).$$

II. Sit $C = \frac{1}{cc}$, et posito $s = \frac{1}{x} + \frac{1}{t}$ fit

$$dt + \frac{2tdx}{x} + dx = \frac{ttdx}{cc}$$

cui particulariter satisfacit $t = c + \frac{cc}{x}$; vt fit

$$s = \frac{cc+cx+xx}{cx(c+x)} \text{ et } \frac{ds}{dx} + \frac{1}{xx} = \frac{1}{(c+x)^2}.$$

atque huius aequationis,

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2}{(c+x)^2}z$$

integrale fit

$$z = \frac{cc+cx+xx}{cx(c+x)}v + \left(\frac{dv}{dx}\right).$$

Ad integrale autem pro t completum inueniendum statuatur

$$t = c + \frac{cc}{x} + \frac{1}{u}$$

fietque

$$du + \frac{2udx}{c} + \frac{dx}{cc} = 0 \text{ seu } dx = \frac{-ccdu}{1+2cu}$$

hinc

$$x = b - \tfrac{c}{2}l(1+2cu)$$

ergo

$$u = \frac{e^{\frac{2(b-x)}{c}} - 1}{2c}$$

vnde

vnde

$$t = c + \frac{cc}{x} + \frac{2c}{e^{\frac{2(b-x)}{c}} - 1} \text{ et}$$

$$s = \frac{1}{x} + \frac{x\left(e^{\frac{2(b-x)}{c}} - 1\right)}{c\left((c+x)e^{\frac{2(b-x)}{c}} + c - x\right)}$$

atque

$$\frac{ds}{dx} + \frac{1}{xx} = \frac{-dt}{ttdx} = \frac{1}{tt}\left(1 + \frac{2t}{x} - \frac{tt}{cc}\right) = \frac{1}{tt}\left(\frac{cc}{xx} - \frac{4e^{u}}{(e^{u}-1)^2}\right).$$

Scholion.

359. Quoniam ſupra inuenimus hanc aequationem:

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) - \frac{i(i+1)}{xx}v$$

integrationem admittere, quippe qui caſus oritur ex generali forma (354.) ſumto $m = 0$, erit problemate huc translato

$$ds - ssdx + \left(f + \frac{i(i+1)}{xx}\right)dx = 0,$$

hincque inuenta quantitate s huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \left(2f + \frac{i(i+1)}{xx} - 2ss\right)z$$

integrale erit

$$z = sv + \left(\frac{dv}{dx}\right).$$

I. Quod ſi iam capiamus $f = 0$, erit particulariter $s = \frac{i}{x}$ vel $s = \frac{-i-1}{x}$, vnde quidem aequationis

nis integrabilis forma non mutatur. At facto $s = \frac{i}{x} + \frac{1}{t}$ oritur

$$dt + \frac{2itdx}{x} + dx = 0$$

cuius integrale est

$$x^{2i}t + \frac{1}{2i+1}x^{2i+1} = \frac{g}{2i+1}$$

ideoque

$$s = \frac{ig + (i+1)x^{2i+1}}{x(g - x^{2i+1})},$$

et aequatio integrabilis fit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{(i(i-1)gg + 6i(i+1)gx^{2i+1} + (i+1)(i+2)x^{4i+2})z}{xx(g - x^{2i+1})^2}.$$

II. At non reiecto f fit $s = \frac{i}{x} + u$ fietque

$$-du + \frac{2iudx}{x} + uudx = fdx$$

quae vt in aequationem differentialem secundi gradus facile per seriem resolubilem conuertatur, ponatur

$$u = \sqrt{f} - \frac{i}{x} - \frac{dr}{rdx}$$

et prodit:

$$\frac{ddr}{dx^2} - \frac{2dr\sqrt{f}}{dx} - \frac{i(i+1)r}{xx} = 0$$

fit $\sqrt{f} = a$ et statuatur

$$r = Ax^{i+1} + Bx^{i+2} + Cx^{i+3} + Dx^{i+4} \text{ etc.}$$

ac reperitur:

$$B = \frac{2(i+1)a}{1(2i+2)}A;\ C = \frac{2(i+2)a}{2(2i+3)}B;\ D = \frac{2(i+3)a}{3(2i+4)}C;\ E = \frac{2(i+4)a}{4(2i+5)}D \text{ etc.}$$

quae

quae abrumpitur quoties i eſt numerus integer negatiuus. Sin autem ſtatuatur

$$r = Ax^{-i} + Bx^{1-i} + Cx^{2-i} + Dx^{3-i} \text{ etc.}$$

ſequens relatio naſcitur

$$B = \frac{2ia}{2i}A;\ C = \frac{2(i-1)a}{2(2i-1)}B;\ D = \frac{2(i-2)a}{3(2i-2)}C;\ E = \frac{2(i-3)a}{4(2i-3)}D \text{ etc.}$$

quae abrumpitur quoties i eſt numerus integer poſitiuus.

Problema 58.

360. Propoſita aequatione

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) - \frac{2aa}{\operatorname{cof.}(ax+b)^2}v$$

cuius integrale eſt:

$$v = a \operatorname{tang.}(ax+b).(\pi:(x+y) + \Phi:(x-y))$$
$$+ \pi':(x+y) + \Phi':(x-y)$$

per transformationem hic traditam alias inuenire aequationes eius ope integrabiles.

Solutio.

Ponamus breuitatis gratia angulum $ax+b=\omega$, vt ſit $d\omega = adx$; et ex §. 351. cum ſit $F=1$, $G=0$, $H=\frac{-2aa}{\operatorname{coſ.}\omega^2}$ quaeratur quantitas s ex hac aequatione

$$ds - ssdx + \left(C + \frac{2aa}{\operatorname{coſ.}\omega^2}\right)dx = 0$$

eritque huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \left(\frac{2aa}{\operatorname{coſ.}\omega^2} + \frac{2ds}{dx}\right)z$$

 in-

integrale

$$z = sv + \left(\frac{dv}{dx}\right) \text{ seu}$$

$$z = as\,\text{tang.}\,\omega(\pi:(x+y)+\Phi:(x-y))+s(\pi':(x+y)+\Phi':(x-y))$$
$$+\frac{aa}{\text{cos}.\,\omega^2}(\pi:(x+y)+\Phi:(x-y))+a\,\text{tang.}\,\omega(\pi':(x+y)+\Phi':(x-y))$$
$$+\pi'':(x+y)+\Phi'':(x-y.$$

Totum ergo negotium ad inuentionem quantitatis s reducitur quem in finem ponamus:

$$s = \alpha\,\text{tang.}\,\omega - \frac{du}{u\,dx},$$

fietque

$$\frac{ds}{dx} = \frac{\alpha\alpha}{\text{cos}.\,\omega^2} - \frac{ddu}{u\,dx^2} + \frac{du^2}{uu\,dx^2},$$

et facta substitutione prodit

$$\frac{\alpha\alpha}{\text{cos}.\,\omega^2} - \frac{ddu}{u\,dx^2} + \frac{2\alpha du}{u\,dx}\text{tang.}\,\omega = 0$$
$$-\frac{\alpha\alpha\,\text{sin}.\,\omega^2}{\text{cos}.\,\omega^2}$$
$$+C+\frac{2\alpha a}{\text{cos}.\,\omega^2}.$$

Iam ob

$$-\frac{\alpha\alpha\,\text{sin}.\,\omega^2}{\text{cos}.\,\omega^2} = -\frac{\alpha\alpha}{\text{cos}.\,\omega^2} + \alpha\alpha,$$

sumatur α ita vt fiat

$$-\alpha\alpha + \alpha\alpha + 2a\alpha = 0.$$

Capiatur ergo $\alpha = -a$, vt sit

$$s = -a\,\text{tang.}\,\omega - \frac{du}{u\,dx}$$

et pro quantitate u inuenienda haec habetur aequatio

$$\frac{ddu}{u\,dx^2} + \frac{2a\,du}{u\,dx}\text{tang.}\,\omega + naa = 0$$

posito

posito $C = -aa - nau$

seu $\frac{ddu}{d\omega^2} + \frac{2du}{d\omega}\,\text{tang.}\,\omega + nu = 0$

ob $dx = \frac{d\omega}{a}$

cuius resolutio non parum ardua videtur, inter complures autem modos eam tractandi hic ad institutum maxime idoneus videtur. Fingatur:

$$u = A\,\text{cos}.\,\lambda\omega + B\,\text{cos}.\,(\lambda+2)\omega + C\,\text{cos}.\,(\lambda+4)\omega + \text{etc.}$$

eritque

$$\frac{du}{d\omega} = -\lambda A\,\text{sin}.\,\lambda\omega - (\lambda+2)\,B\,\text{sin}.\,(\lambda+2)\,\omega$$
$$-(\lambda+4)C\,\text{sin}.\,(\lambda+4)\,\omega \text{ etc.}$$
$$\frac{ddu}{d\omega^2} = -\lambda\lambda\,A\,\text{cos}.\,\lambda\,\omega - (\lambda+2)^2 B\,\text{cos}.\,(\lambda+2)\,\omega$$
$$-(\lambda+4)^2 C\,\text{cos}.\,(\lambda+4)\,\omega \text{ etc.}$$

et aequatio hac forma repraesentata

$$\frac{2ddu}{d\omega^2}\,\text{cos}.\,\omega + \frac{4du}{d\omega}\,\text{sin}.\,\omega + 2nu\,\text{cos}.\,\omega = 0$$

dabit:

$$0 = -\lambda\lambda A\,\text{cos}.(\lambda-1)\omega - (\lambda+2)^2 B\,\text{cos}.(\lambda+1)\omega - (\lambda+4)^2 C\,\text{cos}.(\lambda+3)\omega \text{ etc.}$$

	$-\lambda\lambda A$	$-(\lambda+2)^2 B$
$-2\lambda A$	$-2(\lambda+2)B$	$-2(\lambda+4)C$
	$+2\lambda A$	$+2(\lambda+2)B$
$+nA$	$+nB$	$+nC$
	$+nA$	$+nB$

vnde

vnde λ ita capi oportet vt fit

$$\lambda\lambda + 2\lambda = n \text{ seu } \lambda = -1 \pm \sqrt{(n+1)},$$

duplexque pro λ habeatur valor. Praeterea vero fecundus terminus ob $n = \lambda\lambda + 2\lambda$ praebet: $B = \frac{\lambda}{\lambda+2}A$ tertius vero commode dat $C = 0$, vnde et fequentes omnes euanefcunt.

Sumamus $n = mm - 1$ vt fit

$$\lambda = -1 \pm m \text{ et } B = -\frac{1 \mp m}{1 \pm m}A;$$

atque integrale completum concludi videtur

$$u = A\left(\operatorname{cof.}(m-1)\omega + \frac{m-1}{m+1}\operatorname{cof.}(m+1)\omega\right)$$
$$+ \mathfrak{A}\left(\operatorname{cof.}(m+1)\omega + \frac{m+1}{m-1}\operatorname{cof.}(m-1)\omega\right)$$

fit

$$A = (m+1)B \text{ et } \mathfrak{A} = (m-1)\mathfrak{B}$$

fiet

$$u = (m+1)(B+\mathfrak{B})\operatorname{cof.}(m-1)\omega + (m-1)(B+\mathfrak{B})\operatorname{cof.}(m+1)\omega$$

vbi cum binae conftantes in vnam coalefcant, hoc integrale tantum eft particulare ex quo autem deinceps completum elici poterit. Cum ergo fit

$$\frac{du}{u\,d\omega} = \frac{-(mm-1)\operatorname{fin.}(m-1)\omega - (mm-1)\operatorname{fin.}(m+1)\omega}{(m+1)\operatorname{cof.}(m-1)\omega + (m-1)\operatorname{cof.}(m+1)\omega} \text{ et}$$

$$\frac{z}{a} = -\operatorname{tang.}\omega + \frac{(mm-1)(\operatorname{fin.}(m-1)\omega + \operatorname{fin.}(m+1)\omega)}{(m+1)\operatorname{cof.}(m-1)\omega + (m-1)\operatorname{cof.}(m+1)\omega}$$

pro aequatione:

$$\frac{dz}{a\,d\omega} - \frac{zz}{aa} - mm + \frac{2}{\operatorname{cof.}\omega^2} = 0$$

ob $C = -(n+1)aa = -mmaa$.

Illud

Illud autem integrale inuentum ad hanc formam reducitur

$$\frac{s}{a} = -\mathrm{tang.}\,\omega + \frac{(mm-1)\,\mathrm{tang.}\,m\omega}{m+\mathrm{tang.}\,m\omega\,\mathrm{tang.}\,\omega}$$

quae expreſſio ſubſtituta illi aequationi egregie ſatisfacere deprehenditur. Scribamus eius loco Θ, ac ponamus $\frac{s}{a} = \Theta + \frac{1}{t}$ pro integrali completo eliciendo, prodibitque:

$$-\frac{dt}{tt\,d\omega} - \frac{2\Theta}{t} - \frac{1}{tt} = 0 \text{ ſeu}$$

$$dt + 2\Theta t\,d\omega + d\omega = 0.$$

Erat autem modo ante

$$\Theta = \frac{s}{a} = -\mathrm{tang.}\,\omega - \frac{du}{u\,d\omega},$$

vnde

$$\int \Theta\,d\omega = l\,\mathrm{cof}\,\omega - lu \text{ et } e^{2\int\Theta d\omega} = \frac{\mathrm{cof.}\,\omega^2}{uu},$$

qui eſt multiplicator pro illa aequatione, ſicque fit

$$\frac{t\,\mathrm{cof.}\,\omega^2}{uu} = C - \int \frac{d\omega\,\mathrm{cof.}\,\omega^2}{uu}$$

at eſt

$$u = 2m\,\mathrm{cof.}\,m\omega\,\mathrm{cof.}\,\omega + 2\,\mathrm{ſin.}\,m\omega\,\mathrm{ſin.}\,\omega,$$

ideoque:

$$\frac{t}{(m\,\mathrm{cof.}\,m\omega + \mathrm{ſin.}\,m\omega\,\mathrm{tang.}\,\omega)^2} = A - \int \frac{d\omega}{(m\,\mathrm{cof.}\,m\omega + \mathrm{ſin.}\,m\omega\,\mathrm{tang.}\,\omega)^2}$$

cuius poſtremi membri integrale deprehenditur

$$\frac{-m\,\mathrm{tang.}\,m\omega + \mathrm{tang.}\,\omega}{m(mm-1)(m+\mathrm{tang.}\,m\omega\,\mathrm{tang.}\,\omega)} = \frac{-m\,\mathrm{ſin.}\,m\omega + \mathrm{tang.}\,\omega\,\mathrm{cof.}\,m\omega}{m(mm-1)(m\,\mathrm{cof.}\,m\omega + \mathrm{ſin.}\,m\omega\,\mathrm{tang.}\,\omega)}$$

ita vt ſit

$$\frac{t}{(m\,\mathrm{cof.}\,m\omega + \mathrm{ſin.}\,m\omega\,\mathrm{tang.}\,\omega)^2} = A + \frac{\mathrm{cof.}\,m\omega\,\mathrm{tang.}\,\omega - m\,\mathrm{ſin.}\,m\omega}{m(mm-1)(m\,\mathrm{cof.}\,m\omega + \mathrm{ſin.}\,m\omega\,\mathrm{tang.}\,\omega)}$$

 ſeu

feu

$$\frac{z}{t}=\frac{m(mm-1)}{(C(m\,\text{cof.}\,m\omega+\text{fin.}\,m\omega\,\text{tang.}\,\omega)+\text{cof.}\,m\omega\,\text{tang.}\,\omega-m\,\text{fin.}\,m\omega)(m\,\text{cof.}\,m\omega+\text{fin.}\,m\omega\,\text{tang.}\,\omega)}$$

cui addatur

$$\Theta=-\text{tang.}\,\omega+\frac{(mm-1)\,\text{fin.}\,m\omega}{m\,\text{cof.}\,m\omega+\text{fin.}\,m\omega\,\text{tang.}\,\omega}$$

vt prodeat $\frac{s}{a}$ eritque

$$\frac{s}{a}=-\text{tang.}\,\omega+\frac{(mm-1)(C\,\text{fin.}\,m\omega+\text{cof.}\,m\omega)}{C(m\,\text{cof.}\,m\omega+\text{fin.}\,m\omega\,\text{tang.}\,\omega)+\text{cof.}\,m\omega\,\text{tang.}\,\omega-m\,\text{fin.}\,m\omega}$$

feu

$$\frac{s}{a}=\frac{(mm-1-\text{tang.}\,\omega^2)(C\,\text{fin.}\,m\omega+\text{cof.}\,m\omega)-m\,\text{tang.}\,\omega(C\,\text{cof.}\,m\omega-\text{fin.}\,m\omega)}{C(m\,\text{cof.}\,m\omega+\text{fin.}\,m\omega\,\text{tang.}\,\omega)+\text{cof.}\,m\omega\,\text{tang.}\,\omega-m\,\text{fin.}\,m\omega}.$$

Coroll. 1.

361. Hic praecipue notandum eft huius aequationis

$$\frac{ddu}{d\omega^2}+\frac{2du}{d\omega}\,\text{tang.}\,\omega+(mm-1)u=0$$

integrale particulare effe

$$u=m\,\text{cof.}\,\omega\,\text{cof.}\,m\omega+\text{fin.}\,m\omega\,\text{fin.}\,\omega$$

aliud vero integrale particulare reperitur fimili modo:

$$u=m\,\text{fin}\,m\omega\,\text{cof.}\,\omega-\text{cof.}\,m\omega\,\text{fin.}\,\omega\,,$$

vnde concluditur completum

$$u=A\,(m\,\text{cof.}\,m\omega\,\text{cof.}\,\omega+\text{fin.}\,m\omega\,\text{fin.}\,\omega)$$
$$+B\,(m\,\text{fin.}\,m\omega\,\text{cof.}\,\omega-\text{cof.}\,m\omega\,\text{fin.}\,\omega).$$

Coroll. 2.

362. Si hic ponatur

$$A=C\,\text{cof.}\,\alpha \text{ et } B=-C\,\text{fin.}\,\alpha$$

hoc

hoc integrale completum ad hanc formam redigitur:

$$u = C(m\,\mathrm{cos.}(m\omega + \alpha)\,\mathrm{cos.}\,\omega + \mathrm{sin.}(m\omega + \alpha)\,\mathrm{sin.}\,\omega)$$

quod quidem ex integrali particulari primum invento statim concludi potuisset, cum ibi loco anguli $m\omega$ scribere liceat $m\omega + \alpha$.

Coroll. 3.

363. Hinc multo facilius reperitur valor

$$\frac{s}{a} = -\,\mathrm{tang.}\,\omega - \frac{du}{u\,d\omega}$$

cum enim sit

$$\frac{du}{d\omega} = -C(mm-1)\,\mathrm{sin.}(m\omega + \alpha)\,\mathrm{cos.}\,\omega$$

erit

$$\frac{s}{a} = -\,\mathrm{tang.}\,\omega + \frac{(mm-1)\,\mathrm{sin.}(m\omega+\alpha)\,\mathrm{cos.}\,\omega}{m\,\mathrm{cos.}(m\omega+\alpha)\,\mathrm{cos.}\,\omega + \mathrm{sin.}(m\omega+\alpha)\,\mathrm{sin.}\,\omega}$$

hincque

$$\frac{ds}{a\,d\omega} = \frac{ds}{aa\,dx} = \frac{-1}{\mathrm{cos.}\,\omega^2} + \frac{(mm-1)(m^2\,\mathrm{cos.}\,\omega^2 - \mathrm{sin.}(m\omega+\alpha)^2)}{(m\,\mathrm{cos.}(m\omega+\alpha)\,\mathrm{cos.}\,\omega + \mathrm{sin.}(m\omega+\alpha)\,\mathrm{sin.}\,\omega)^2}$$

et aequatio, cuius integrationem inuenimus, erit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2(mm-1)aa(m^2\,\mathrm{cos.}\,\omega^2 - \mathrm{sin.}(m\omega+\alpha)^2)}{(m\,\mathrm{cos.}(m\omega+\alpha)\,\mathrm{cos.}\,\omega + \mathrm{sin.}(m\omega+\alpha)\,\mathrm{sin.}\,\omega)^2}\,z$$

eiusque integrale colligitur

$$z = \frac{maa(m\,\mathrm{sin.}(m\omega+\alpha)\,\mathrm{sin.}\,\omega + \mathrm{cos.}(m\omega+\alpha)\,\mathrm{cos.}\,\omega)}{m\,\mathrm{cos.}(m\omega+\alpha)\,\mathrm{cos.}\,\omega + \mathrm{sin.}(m\omega+\alpha)\,\mathrm{sin.}\,\omega}\,(\Pi:(x+y) + \Phi:(x-y))$$
$$+ \frac{(mm-1)a\,\mathrm{sin.}(m\omega+\alpha)\,\mathrm{cos.}\,\omega}{m\,\mathrm{cos.}(m\omega+\alpha)\,\mathrm{cos.}\,\omega + \mathrm{sin.}(m\omega+\alpha)\,\mathrm{sin.}\,\omega}\,(\Pi':(x+y) + \Phi':(x-y))$$
$$+ (\Pi'':(x+y) + \Phi'':(x-y))$$

existente $\omega = ax + b$.

Scho-

Scholion 1.

364. Omnino memoratu digna est integratio huius aequationis

$$\frac{ddu}{d\omega^2} + \frac{2du}{d\omega}\,\text{tang.}\,\omega + (mm-1)u = 0,$$

vnde occasionem carpo hanc aequationem generaliorem tractandi:

$$\frac{ddu}{d\omega^2} + \frac{2fdu}{d\omega}\,\text{tang.}\,\omega + gu = 0$$

quam primum obseruo posito

$$\frac{du}{u} = -(2f+1)d\omega\,\text{tang.}\,\omega + \frac{dv}{v}$$

vt sit

$$u = \text{cos.}\,\omega^{2f+1}\,v$$

abire in hanc formam:

$$\frac{ddv}{d\omega^2} - \frac{2(f+1)dv}{d\omega}\,\text{tang.}\,\omega + (g-2f-1)v = 0$$

ita vt si illa integrabilis existat casu $f=n$, integrabilis quoque sit casu $f=-n-1$. Iam pro illa aequatione ponatur

$$u = A\,\text{sin.}\,\lambda\omega + B\,\text{sin.}(\lambda+2)\omega + C\,\text{sin}\,(\lambda+4)\omega + D\,\text{sin.}(\lambda+6)\omega + \text{etc.}$$

et facta substitutione in aequatione:

$$\frac{2ddu}{d\omega^2}\text{cos.}\,\omega + \frac{4fdu}{d\omega}\text{sin.}\,\omega + 2gu\,\text{cos.}\,\omega = 0$$

repe-

reperitur

$$\begin{array}{llll}
-\lambda\lambda A \,\text{fi.}(\lambda-1)\omega & -(\lambda+2)^2 B\,\text{fi.}(\lambda+1)\omega & -(\lambda+4)^2 C\,\text{fi.}(\lambda+3)\omega & -(\lambda+6)^2 D\,\text{fi.}(\lambda+5)\omega \\
-2\lambda A f & -\lambda\lambda A & -(\lambda+2)^2 B & -(\lambda+4)^2 C \\
+Ag & +2\lambda A f & +2(\lambda+2)Bf & +2(\lambda+4)Cf \\
 & -2(\lambda+2)Bf & -2(\lambda+4)Cf & -2(\lambda+6)Df \\
 & +Ag & +Bg & +Cg \\
 & +Bg & +Cg & +Dg.
\end{array}$$

Oportet ergo fit $g = \lambda\lambda + 2\lambda f$ tum vero coefficientes aſſumti ita determinantur:

$B = \frac{\lambda f A}{\lambda+f+1}$; $C = \frac{(\lambda+1)(f-1)B}{2(\lambda+f+2)}$; $D = \frac{(\lambda+2)(f-2)C}{3(\lambda+f+3)}$ etc.

Statuamus ergo $g = mm - ff$, vt fiat $\lambda = m - f$, et aequationes noſtrae ſint

$$\frac{ddu}{d\omega^2} + \frac{2f\,du}{d\omega}\,\text{tang.}\,\omega + (mm - ff)u = 0 \text{ et}$$

$$\frac{ddv}{d\omega^2} - \frac{2(f+1)\,dv}{d\omega}\,\text{tang.}\,\omega + (mm - (f+1)^2)v = 0$$

exiſtente

$$u = v\,\text{cof.}\,\omega^{f+1} \quad \text{feu} \quad v = \frac{u}{\text{cof.}\,\omega^{f+1}}.$$

Quoniam nunc ſeries noſtra abrumpitur, quoties eſt f numerus integer, percurramus caſus ſimpliciores.

I Sit $f = 0$, erit

$$\lambda = m \text{ et } B = 0,\ C = 0 \text{ etc.}$$

ideoque

$$u = A\,\text{fin.}\,m\omega \text{ et } v = \frac{A\,\text{fin.}\,m\omega}{\text{cof.}\,\omega}.$$

II. Sit $f=1$, erit

$$\lambda = m-1 \text{ et } B=\frac{(m-1)A}{m+1}, \; C=0 \text{ etc.}$$

ergo

$$\frac{u}{a}=(m+1)\sin.(m-1)\omega+(m-1)\sin.(m+1)\omega \text{ et } v=\frac{u}{\cos\omega^3}$$

seu $\frac{u}{2a}=m\sin.m\omega\cos.\omega-\cos.m\omega\sin.\omega.$

III. Sit $f=2$, erit $\lambda=m-2$, et

$$B=\frac{2(m-2)A}{m+1};\; C=\frac{(m-1)B}{2(m+2)}=\frac{(m-1)(m-2)A}{(m+1)(m+2)};\; D=0 \text{ etc.}$$

hinc

$$\frac{u}{a}=(m+1)(m+2)\sin.(m-2)\omega+2(m-2)(m+2)\sin.m\omega$$
$$+(m-1)(m-2)\sin.(m+2)\omega \text{ indeque } v=\frac{u}{\cos.\omega^5}$$

seu $\frac{u}{2a}=(mm+2)\sin.m\omega\cos.2\omega+2(mm-4)\sin.m\omega$
$-3m\cos.m\omega\sin.2\omega.$

IV. Sit $f=3$, erit $\lambda=m-3$ et

$$B=\frac{3(m-3)A}{m+1};\; C=\frac{2(m-2)B}{2(m+2)} \text{ et } D=\frac{(m-1)C}{3(m+3)};\; E=0 \text{ etc.}$$

Ergo

$$\frac{u}{a}=+(m+1)(m+2)(m+3)\sin.(m-3)\omega+3(m+2)(mm-9)\sin.(m-1)\omega$$
$$+(m-1)(m-2)(m-3)\sin.(m+3)\omega+3(m-2)(mm-9)\sin.(m+1)\omega$$

existente $v=\frac{u}{\cos.\omega^7}.$

V.

V. Sit $f=4$ erit $\lambda=m-4$ ac reperitur:

$$\frac{u}{v}=+(m+1)(m+2)(m+3)(m+4)\text{fi.}(m-4)\omega+4(m+2)(m+3)(mm-16)\text{fi.}(m-2)\omega$$
$$+(m-1)(m-2)(m-3)(m-4)\text{fi.}(m+4)\omega+4(m-2)(m-3)(mm-16)\text{fi.}(m+2)\omega$$
$$+6(mm-9)(mm-16)\text{fi.}m\omega$$

existente $v=\frac{u}{\text{cof.}\,\omega^{9}}$

vnde ratio progressionis per se est manifesta. Notari autem conuenit si posuissemus:

$$u=A\,\text{cof.}\,\lambda\omega+B\,\text{cof.}\,(\lambda+2)\omega+C\,\text{cof.}\,(\lambda+4)\omega+\text{etc.}$$

easdem coefficientium determinationes prodituras fuisse ex qua hi duo valores coniuncti integrale completum exhibebunt: quod etiam ex forma inuenta colligitur, si modo loco anguli $m\omega$ generalius scribatur $m\omega+\alpha$.

Scholion 2.

365. Pluribus autem aliis modis eadem aequatio

$$\frac{ddu}{d\omega^2}+\frac{2f\,du}{d\omega}\,\text{tang.}\,\omega+gu=0$$

tractari et eius integrale per series exprimi potest, vnde alii casus integrabilitatis obtineantur. Ad hoc primum notetur posito $u=\text{fin.}\,\omega^\lambda$ fore

$$\frac{du}{d\omega}=\lambda\,\text{fin}\,\omega^{\lambda-1}\,\text{cof}\,\omega \text{ hincque}$$

$$\frac{du}{d\omega}\,\text{tang.}\,\omega=\lambda\,\text{fin.}\,\omega^\lambda \text{ et}$$

$$\frac{ddu}{d\omega^2}=\lambda(\lambda-1)\,\text{fin.}\,\omega^{\lambda-2}\,\text{cof.}\,\omega^2-\lambda\,\text{fin.}\,\omega^\lambda=\lambda(\lambda-1)\,\text{fin.}\,\omega^{\lambda-2}$$
$$-\lambda\lambda\,\text{fin.}\,\omega^\lambda.$$

Hinc

Hinc si ponamus:

$$u = A\,\text{sin.}\,\omega^{\lambda} + B\,\text{sin.}\,\omega^{\lambda+2} + C\,\text{sin.}\,\omega^{\lambda+4} + D\,\text{sin.}\,\omega^{\lambda+6} + \text{etc.}$$

facta substitutione adipiscimur:

$$\begin{array}{llll} 0 = \lambda(\lambda-1)A\,\text{si.}\,\omega^{\lambda-2} & + (\lambda+2)(\lambda+1)B\,\text{si.}\,\omega^{\lambda} & + (\lambda+4)(\lambda+3)C\,\text{si.}\,\omega^{\lambda+2} & + \text{etc.} \\ \quad -\lambda\lambda A & -(\lambda+2)^2 B & & \\ \quad +2\lambda f A & +2(\lambda+2)fB & & \\ \quad +gA & +gB & & \end{array}$$

vnde sumi oportet vel $\lambda = 0$ vel $\lambda = 1$, tum vero erit

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+1)(\lambda+2)}A\,;\quad C = \frac{(\lambda+2)^2 - 2(\lambda+2)f - g}{(\lambda+3)(\lambda+4)}B \text{ etc.}$$

hinc duo casus euolui conuenit

$\lambda = 0$	$\lambda = 1$
$B = \frac{-g}{1\cdot 2}A$	$B = \frac{1-2f-g}{2\cdot 3}A$
$C = \frac{4-4f-g}{3\cdot 4}B$	$C = \frac{9-6f-g}{4\cdot 5}B$
$D = \frac{16-8f-g}{5\cdot 6}C$	$D = \frac{25-10f-g}{6\cdot 7}C$
$E = \frac{36-12f-g}{7\cdot 8}D$	$E = \frac{49-14f-g}{8\cdot 9}D$
etc.	etc.

Integratio ergo succedit, quoties fuerit $g = ii - 2if$ denotante i numerum integrum positiuum. Quare cum posito $u = v\,\text{cos.}\,\omega^{2f+1}$ aequatio transformata sit

$$\frac{ddv}{d\omega^2} - \frac{2(f+1)dv}{d\omega}\,\text{tang.}\,\omega + (g - 2f - 1)v = 0$$

haec ideoque et illa erit integrabilis quoties fuerit:

$$g = (i+1)^2 + 2(i+1)f$$

quos

quos binos caſus ita vno complecti licet, vt integratio ſuccedat dum ſit $g = ii \pm 2if$.

Scholion 3.

366. Eidem aequationi adhuc inhaerens, cum poſito $u = \cos.\omega^{\lambda}$, ſit

$$\frac{du}{d\omega} = -\lambda \cos.\omega^{\lambda-1} \sin.\omega,$$

ideoque

$$\frac{du}{d\omega}\,\mathrm{tang}.\omega = -\lambda \cos.\omega^{\lambda-2} + \lambda \cos.\omega^{\lambda} \text{ et}$$

$$\frac{ddu}{d\omega^2} = \lambda(\lambda-1)\cos.\omega^{\lambda-2} - \lambda\lambda \cos.\omega^{\lambda},$$

ſtatuo:

$$u = A\cos.\omega^{\lambda} + B\cos.\omega^{\lambda+2} + C\cos.\omega^{\lambda+4} + D\cos.\omega^{\lambda+6} + \text{etc.}$$

et facta ſubſtitutione orietur:

$$\begin{array}{llll}
0 = \lambda(\lambda-1)A\cos.\omega^{\lambda-2} & + (\lambda+2)(\lambda+1)B\cos.\omega^{\lambda} & + (\lambda+4)(\lambda+3)C\cos.\omega^{\lambda+2} & + \text{etc.} \\
\quad -2\lambda f A & -\lambda\lambda A & -(\lambda+2)^2 B & \\
 & -2(\lambda+2)fB & -2(\lambda+4)fC & \\
 & +2\lambda f A & +2(\lambda+2)fB & \\
 & +gA & +gB. &
\end{array}$$

Oportet ergo ſit vel $\lambda = 0$ vel $\lambda = 2f+1$, tum vero

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+2)(\lambda+1-2f)}A;\; C = \frac{(\lambda+2)^2 - 2(\lambda+2)f - g}{(\lambda+4)(\lambda+3-2f)}B \text{ etc.}$$

et ambo casus ita se habebunt:

$\lambda = 0$	$\lambda = 2f + 1$
$B = \frac{-g}{2(1-2f)} A$	$B = \frac{1+2f-g}{2(2f+3)} A$
$C = \frac{4-4f-g}{4(3-2f)} B$	$C = \frac{9+6f-g}{4(2f+5)} B$
$D = \frac{16-8f-g}{6(5-2f)} C$	$D = \frac{25-10f-g}{6(2f+7)} C$
etc.	etc.

Ex priori integratio succedit si $g = 4ii - 4if$, ex posteriori si $g = (2i+1)^2 + 2(2i+1)f$, qui casus cum iis, qui ex transformata nascuntur iuncti, eodem redeunt ac in §. praec. inuenti. Omnes ergo hactenus inuenti integrabilitatis casus huc reuocantur vt posito $g = mm - ff$, fit vel $f = \pm i$ vel $m = i \pm f$, hoc est vel $f = \pm i$ vel $f = \pm i \pm m$. Ceterum hi posteriores casus etiam ex prima resolutione (364.) sequuntur, vbi series quoque abrumpitur si $\lambda = -i$, ideoque $g = mm - ff = ii - 2if$ ergo $i - f = \pm m$, et transformatione in subsidium vocata $f = \pm i \pm m$. Contra vero casus primo inuenti in resolutionibus posterioribus non occurrunt.

Problema 59.

367. Concessa huius aequationis integratione

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

inuenire aequationem huius formae:

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz$$

pro-

pro qua fit

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv$$

vbi F, G, H; P, Q, R; et r, s funt functiones ipfius x tantum.

Solutio.

Cum fit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{d^4v}{dx^2dy^2}\right) + r\left(\frac{d^3v}{dxdy^2}\right) + s\left(\frac{ddv}{dy^2}\right), \text{ ob}$$

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv \text{ erit}$$

$$\left(\frac{d^3v}{dxdy^2}\right) = F\left(\frac{d^3v}{dx^3}\right) + \frac{dF}{dx}\left(\frac{ddv}{dx^2}\right) + \frac{dG}{dx}\left(\frac{dv}{dx}\right) + \frac{dH}{dx}v \text{ etc.}$$
$$\qquad\qquad\qquad\qquad + G \qquad\qquad + H$$

$$\left(\frac{d^4v}{dx^2dy^2}\right) = F\left(\frac{d^4v}{dx^4}\right) + \frac{2dF}{dx}\left(\frac{d^3v}{dx^3}\right) + \frac{ddF}{dx^2}\left(\frac{ddv}{dx^2}\right) + \frac{ddG}{dx^2}\left(\frac{dv}{dx}\right) + \frac{ddH}{dx^2}v$$
$$\qquad\qquad + G \qquad + \frac{2dG}{dx} \qquad + \frac{2dH}{dx}$$
$$\qquad\qquad\qquad\qquad + H.$$

Deinde vero ob

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv \text{ fit}$$

$$\left(\frac{dz}{dx}\right) = \left(\frac{d^3v}{dx^3}\right) + r\left(\frac{ddv}{dx^2}\right) + \frac{dr}{dx}\left(\frac{dv}{dx}\right) + \frac{ds}{dx}v \text{ et}$$
$$\qquad\qquad\qquad\qquad + s$$

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{d^4v}{dx^4}\right) + r\left(\frac{d^3v}{dx^3}\right) + \frac{2dr}{dx}\left(\frac{ddv}{dx^2}\right) + \frac{ddr}{dx^2}\left(\frac{dv}{dx}\right) + \frac{dds}{dx^2}v$$
$$\qquad\qquad\qquad + s \qquad\qquad + \frac{2ds}{dx}.$$

His iam fubftitutis neceffe eft, vt omnes termini affecti per $\left(\frac{d^4v}{dx^4}\right)$, $\left(\frac{d^3v}{dx^3}\right)$, $\left(\frac{ddv}{dx^2}\right)$, $\left(\frac{dv}{dx}\right)$ et v feor-

fim

ſim euaneſcant vnde ſequentes reſultant aequationes:

$$\begin{array}{c|l}
\text{ex} & \\
\left(\frac{d^4 v}{d x^4}\right) & \text{I. } F = P \\
\left(\frac{d^3 v}{d x^3}\right) & \text{II. } G + \frac{2 dF}{dx} + Fr = Pr + Q \\
\left(\frac{dd v}{d x^2}\right) & \text{III. } H + \frac{2dG}{dx} + \frac{ddF}{dx^2} + Gr + \frac{rdF}{dx} + Fs = P\left(s + \frac{2dr}{dx}\right) + Qr + R \\
\left(\frac{d v}{d x}\right) & \text{IV. } \frac{2dH}{dx} + \frac{ddG}{dx^2} + Hr + \frac{rdG}{dx} + Gs = P\left(\frac{2ds}{dx} + \frac{ddr}{dx^2}\right) + Q\left(s + \frac{dr}{dx}\right) + Rr \\
v & \text{V. } \frac{ddH}{dx^2} + \frac{rdH}{dx} + Hs = P\frac{dds}{dx^2} + Q\frac{ds}{dx} + Rs.
\end{array}$$

Ex prima fit $P = F$ ex ſecunda $Q = G + \frac{2 dF}{dx}$, et tertia $R = H + \frac{2 dG}{dx} + \frac{ddF}{dx^2} - \frac{rdF - 2Fdr}{dx}$, qui valores in binis vltimis ſubſtituti praebent:

$$\frac{2dH}{dx} + \frac{ddG}{dx^2} - \frac{rdG - Gdr}{dx} - \frac{rddF}{dx^2} - \frac{2dFdr}{dx^2} - \frac{2sdF - 2Fds}{dx}$$
$$+ \frac{rrdF + 2Frdr}{dx} - \frac{Fddr}{dx^2} = 0 \quad \text{et}$$

$$\frac{ddH}{dx^2} + \frac{rdH}{dx} - \frac{sddF - 2dFds - Fdds}{dx^2} - \frac{2sdG - Gds}{dx} + \frac{s(rdF + 2Fdr)}{dx} = 0$$

quarum illa ſponte eſt integrabilis praebens:

$$2H + \frac{dG}{dx} - Gr - \frac{rdF - Fdr}{dx} - 2Fs + Frr = A,$$

deinde binis illis aequationibus ita repraeſentatis

$$-\frac{dd.Fr}{dx^2} - \frac{2d.Fs}{dx} + \frac{d.Frr}{dx} + \frac{ddG}{dx^2} - \frac{d.Gr}{dx} + \frac{2dH}{dx} = 0$$

$$-\frac{dd.Fs}{dx^2} + \frac{s}{r}.\frac{d.Frr}{dx} - \frac{2sdG - Gds}{dx} + \frac{rdH}{dx} + \frac{ddH}{dx^2} = 0$$

vel adeo hoc modo

$$\frac{dd.(G - Fr)}{dx} - d.r(G - Fr) + 2d.(H - Fs) = 0$$

$$\frac{dd.(H - Fs)}{dx} + 2Fsdr + rsdF - Gds - 2sdG + rdH = 0$$

vltima

vltima vero ita repraesentari potest

$$\frac{dd.(H-Fs)}{dx}-2sd.(G-Fr)-ds(G-Fr)+rd.(H-Fs)=0.$$

Quod si iam prior per $H-Fs$ haec vero per $-(G-Fr)$ multiplicetur summa fit

$$\begin{aligned}&\frac{(H-Fs)dd.(G-Fr)-(G-Fr)dd.(H-Fs)}{dx} &&-(G-Fr)(H-Fs)dr=0\\ &+2(H-Fs)d.(H-Fs) &&-r(H-Fs)d.(G-Fr)\\ &+2s(G-Fr)d.(G-Fr)+(G-Fr)^2ds &&-r(G-Fr)d.(H-Fs)\end{aligned}$$

cuius integrale manifesto est

$$\begin{aligned}\frac{(H-Fs)d.(G-Fr)-(G-Fr)d.(H-Fs)}{dx} &+(H-Fs)^2+(G-Fr)^2s\\ &-(G-Fr)(H-Fs)r=B\end{aligned}$$

integrale autem prius inuentum est

$$\frac{d.(G-Fr)}{dx}-(G-Fr)r+2(H-Fs)=A$$

quae per $H-Fs$ multiplicata et ab illa subtracta relinquit

$$-\frac{(G-Fr)d.(H-Fs)}{dx}-(H-Fs)^2+(G-Fr)^2s=B-A(H-Fs)$$

sicque habentur duae aequationes simpliciter differentiales ex quibus binas quantitates r et s definiri oportet, quibus cognitis etiam functiones P, Q et R innotescunt.

Coroll. 1.

368. Si sit $F=1$, $G=0$ et $H=0$, aequationes inuentae, erunt

$$-\frac{dr}{dx}+rr-2s=a \text{ et } \frac{sdr-rds}{dr}+ss=b;$$

 vnde

vnde dx eliminando fit

$$\frac{rds - sdr}{ar} = \frac{b - ss}{a + 2s - rr} \text{ seu } \frac{rds}{dr} = \frac{b + as + ss - rrs}{a + 2s - rr}$$

cuius resolutio in genere vix suscipienda videtur. Sumtis autem constantibus $a = 0$ et $b = 0$ aequatio $\frac{rds}{dr} = \frac{ss - rrs}{2s - rr}$ posito $s = rrt$ transit in:

$$\frac{rdt + 2tdr}{dr} = \frac{tt - t}{2t - 1} \text{ seu } \frac{rdt}{dr} = \frac{-3tt + t}{2t - 1},$$

vnde fit

$$\frac{dr}{r} = \frac{dt(1 - 2t)}{t(3t - 1)} = \frac{-dt}{t} + \frac{dt}{3t - 1}, \text{ et}$$

$$r = \frac{\alpha \sqrt[3]{(3t - 1)}}{t} \text{ hinc}$$

$$s = \frac{\alpha\alpha \sqrt[3]{(3t - 1)^2}}{t}.$$

Coroll. 2.

369. Pro eodem casu singulari ponamus $3t - 1 = u^3$, vt fiat

$$r = \frac{3\alpha u}{1 + u^3} \text{ et } s = \frac{3\alpha\alpha uu}{1 + u^3}.$$

Iam ob $a = 0$ est

$$dx = \frac{dr}{rr - 2s} = \frac{dr}{rr(1 - 2t)} = \frac{3dr}{rr(1 - 2u^3)} \text{ at}$$

$$\frac{dr}{rr} = \frac{du}{3\alpha uu} - \frac{2udu}{3\alpha} = \frac{du(1 - 2u^3)}{3\alpha uu}$$

ita vt sit

$$dx = \frac{du}{\alpha uu} \text{ hincque}$$

$$\frac{1}{u} = \beta - \alpha x \text{ et } u = \frac{1}{\beta - \alpha x};$$

vbi

vbi quidem salua generalitate sumi potest

$$\beta = 0 \quad \text{et} \quad u = \frac{-1}{a x},$$

vnde fit

$$r = \frac{-3xx}{x^3 + c^3} \text{ facto}$$

$$\alpha = -\frac{1}{c} \quad \text{et} \quad s = \frac{3x}{x^3 + c^3}.$$

Tandem ergo colligitur

$$P = 1, \; Q = 0 \text{ et } R = -\frac{2dr}{dx} = -\frac{6x(2c^3 - x^3)}{(c^3 + x^3)^2}.$$

Coroll. 3.

370. Proposita ergo aequatione $\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right)$, cuius integrale est

$$v = \Gamma:(x+y) + \Delta:(x-y),$$

huius aequationis integrale assignari poterit:

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{6x(2c^3 - x^3)}{(c^3 + x^3)^2} z$$

est enim

$$z = \left(\frac{ddv}{dx^2}\right) - \frac{3xx}{c^3 + x^3}\left(\frac{dv}{dx}\right) + \frac{3x}{c^3 + x^3} v.$$

Scholion 1.

371. Haec pro casu

$$F = 1, \; G = 0 \text{ et } H = 0$$

multo facilius atque generalius computari possunt pro quocunque valore quantitatis a, dum sit $b = 0$, tum

tum enim altera aequatio statim dat

$$dx = \frac{r\,ds - s\,dr}{ss} \text{ hincque}$$

$$x = \frac{-r}{s} \text{ et } s = \frac{-r}{x},$$

ex quo prima aequatio hanc induit formam

$$\frac{dr}{dx} - rr - \frac{2r}{x} + a = 0.$$

Ponamus $r = \frac{a}{t}$, fiet

$$dt + \frac{2t\,dx}{x} - tt\,dx + a\,dx = 0$$

cui particulariter satisfacit

$$t = \sqrt{a} + \frac{1}{x}.$$

Statuatur ergo

$$t = \sqrt{a} + \frac{1}{x} + \frac{1}{u}$$

ac prodit:

$$du + dx + 2u\,dx\sqrt{a} = 0,$$

quae per $e^{2x\sqrt{a}}$ multipl. et integrata praebet:

$$e^{2x\sqrt{a}}u + \frac{1}{2\sqrt{a}}e^{2x\sqrt{a}} = \frac{n}{2\sqrt{a}};$$

ideoque

$$\frac{1}{u} = \frac{2e^{2x\sqrt{a}}\sqrt{a}}{n - e^{2x\sqrt{a}}} = \frac{2\sqrt{a}}{ne^{-2x\sqrt{a}} - 1} \text{ et}$$

$$t = \frac{1}{x} + \frac{ne^{-2x\sqrt{a}} + 1}{ne^{-2x\sqrt{a}} - 1}\sqrt{a} = \frac{1}{x} + \frac{n + e^{2x\sqrt{a}}}{n - e^{2x\sqrt{a}}}\sqrt{a} \text{ et}$$

$$r = \frac{ax(n - e^{2x\sqrt{a}})}{n(x\sqrt{a} + 1) + e^{2x\sqrt{a}}(x\sqrt{a} - 1)}$$

ac

ac propterea

$$s = \frac{-a(n-e^{2x\sqrt{a}})}{n(x\sqrt{a}+1)+e^{2x\sqrt{a}}(x\sqrt{a}-1)}$$

tum vero poſtremo

$P=-1$, $Q=0$ et $R=-\frac{2\,dr}{dx}=-2rr-\frac{4r}{x}+2a$ ſeu

$$R=\frac{-2a(nn-4naxxe^{2x\sqrt{a}}-2ne^{2x\sqrt{a}}+e^{4x\sqrt{a}})}{(n(x\sqrt{a}+1)+e^{2x\sqrt{a}}(x\sqrt{a}-1))^2}$$

$$=\frac{-2a(n-e^{2x\sqrt{a}})^2+8naaxxe^{2x\sqrt{a}}}{(n(x\sqrt{a}+1)+e^{2x\sqrt{a}}(x\sqrt{a}-1))^2}.$$

Si iam ſumatur a euaneſcens et $n=1+\frac{2}{3}ac^3\sqrt{a}$ formulae ante inuentae reſultant. At ſi a ſit quantitas negatiua puta $a=-m^2$, capiaturque $n=\frac{\alpha\sqrt{-1}+\beta}{\alpha\sqrt{-1}-\beta}$ reperitur

$$r=\frac{-mmx(\beta\cos.mx+\alpha\sin.mx)}{\beta\cos.mx+\alpha\sin.mx-mx(\alpha\cos.mx-\beta\sin.mx)}=\frac{-mmx\cos.(mx+\gamma)}{\cos.(mx+\gamma)-mx\sin.(mx+\gamma)},$$

et $s=\frac{mm\cos.(mx+\gamma)}{\cos.(mx+\gamma)-mx\sin.(mx+\gamma)}$

indeque

$$R=\frac{2mm(\cos.(mx+\gamma)^2+mmxx)}{(\cos.mx+\gamma)-mx\sin.(mx+\gamma))^2}.$$

Quantitas R reducitur ad hanc

$$R=\frac{8naaxx-2a(ne^{-x\sqrt{a}}-e^{x\sqrt{a}})^2}{(n(1+x\sqrt{a})e^{-x\sqrt{a}}-(1-x\sqrt{a})e^{x\sqrt{a}})^2}$$

quae forma ſumto a valde paruo abit in

$$R=\frac{8naaxx-2a(n-1-(n+1)x\sqrt{a}+\frac{(n-1)}{2}axx-\frac{(n+1)}{6}ax^3\sqrt{a}+\text{etc.})^2}{(n-1-\frac{1}{2}(n-1)axx+\frac{1}{3}(n+1)ax^3\sqrt{a})^2}.$$

Statuatur $n = 1 + \beta a \sqrt{a}$, vt fit

$$n - 1 = \beta a \sqrt{a} \text{ et } n + 1 = 2 + \beta a \sqrt{a}$$

erit

$$R = \frac{8naaxx - 2a\left(\beta a\sqrt{a} - 2x\sqrt{a} - \beta aax + \frac{\beta aaxx\sqrt{a}}{2} - \frac{1}{3}ax^3\sqrt{a}\right)^2}{\left(\beta a\sqrt{a} - \frac{1}{2}\beta aaxx\sqrt{a} + \frac{2}{3}ax^3\sqrt{a}\right)^2}$$

vbi numerator fit

$$8aaxx + 8\beta a^3xx\sqrt{a} - 2a(\beta\beta a^3 - 4\beta aax - 2\beta\beta a^4x\sqrt{a}$$
$$+ 4axx + \tfrac{4}{3}aax^4$$

vbi cum termini per aa affecti se destruant, retineantur ii soli qui per a^3 sunt affecti, erit idem in denominatore obseruato:

$$R = \frac{8\beta a^3x - \frac{8}{3}a^3x^4}{a^3(\beta + \frac{2}{3}x^3)^2} = \frac{8x(\beta - \frac{1}{3}x^3)}{(\beta + \frac{2}{3}x^3)^2}$$

quae iam facile ad formam

$$R = \frac{6x(2c^3 - x^3)}{(c^3 + x^3)^2}$$

reducitur sumendo

$$3\beta = 2c^3 \text{ vt sit } \beta = \tfrac{2}{3}c^3.$$

Quare hic casus oritur sumendo a euanescens et

$$n = 1 + \tfrac{2}{3}c^3a\sqrt{a}.$$

Scholion 2.

372. Cum euolutio solutionis inuentae sit difficillima, neque vlla via pateat, quomodo ambae quan-

quantitates incognitae r et s ex binis aequationibus erutis definiri queant, in scientiae incrementum haud parum iuuabit obseruasse, idem problema, per repetitionem transformationis in primo problem. huius capitis quoque solui posse, neque proinde vsu carebit has duas solutiones inter se comparasse. Proposita ergo aequatione

$$\left(\frac{ddv}{dy^2}\right)=F\left(\frac{ddv}{dx^2}\right)+G\left(\frac{dv}{dx}\right)+Hv$$

ponamus primo

$$u=\left(\frac{dv}{dx}\right)+pv,$$

ac p ex hac aequatione determinetur:

$$F\,dp+Gp\,dx-Fpp\,dx+(C-H)dx=0$$

ac tum ista resultabit aequatio

$$\left(\frac{ddu}{dy^2}\right)=F\left(\frac{ddu}{dx^2}\right)+\left(G+\frac{dF}{dx}\right)\left(\frac{du}{dx}\right)+\left(H+\frac{dG}{dx}-\frac{2F\,dp-p\,dF}{dx}\right)u.$$

Nunc pro hac aequatione porro transformando statuamus simili modo

$$z=\left(\frac{du}{dx}\right)+qu,$$

ita vt sit quoque

$$z=\left(\frac{ddv}{dx^2}\right)+(p+q)\left(\frac{dv}{dx}\right)+\left(\frac{dp}{dx}+pq\right)v$$

et quantitate q ex hac aequatione definita

$$F\,dq+\left(G+\frac{dF}{dx}\right)q\,dx-Fqq\,dx+\left(D-H-\frac{dG}{dx}+\frac{2F\,dp+p\,dF}{dx}\right)dx=0$$

orietur haec aequatio:

$$\left(\frac{ddz}{dy^2}\right)=P\left(\frac{ddz}{dx^2}\right)+Q\left(\frac{dz}{dx}\right)+Rz$$

 cuius

cuius quantitates P, Q, R ita se habent:

$$P = F;\quad Q = G + \frac{2dF}{dx} \text{ et}$$

$$R = H + \frac{2dG}{dx} - \frac{2Fdp - pdF}{dx} + \frac{ddF}{dx^2} - \frac{2Fdq - qdF}{dx}.$$

Cum hac ergo solutione [illegible] postremum problema suppeditauit, in quo cum statim posuerimus:

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv$$

erit vtique

$$r = p + q \text{ et } s = \frac{dp}{dx} + pq,$$

vnde quidem statim valores pro P, Q et R manifesto prodeunt iidem. Verum multo minus apparet, si pro r et s isti valores per p et q substituantur, tum istas binas aequationes:

$$\frac{d.(G - Fr)}{dx} - (G - Fr)r + 2(H - Fs) = A \text{ et}$$

$$\frac{(G - Fr)d.(H - Fs)}{dx} + (H - Fs)^2 - (G - Fr)^2 s - A(H - Fs) = B$$

ad eas quas ante inuenimus reduci:

$$\frac{Fdp}{dx} + Gp - Fpp - H + C = 0 \text{ et}$$

$$\frac{Fdq}{dx} + \left(G + \frac{dF}{dx}\right)q - Fqq - H - \frac{dG}{dx} + \frac{2Fdp + pdF}{dx} + D = 0$$

ita vt hae constantes C et D ad illas A et B certam teneant relationem. Interim patet has postremas aequationes multo esse simpliciores, dum prior duas tantum variabiles p et x complectitur, indeque p per x, cuius F, G et H sunt functiones datae, deter-

determinari debet, qua inuenta quantitatem q simili modo ex altera aequatione elici oportet. Verum in ambabus superioribus aequationibus binae variabiles r et s ita inter se sunt permixtae, vt nulla me-duas tantum variabiles perueniendi habeatur. Cum igitur certum sit priores solutu difficillimas ad posteriores multo faciliores ope substitutionum assignatarum perduci posse, sine dubio methodus hanc reductionem efficiendi haud contemnenda subsidia in Analysin esse allatura videtur.

Scholion 3.

373. Cum adeo consensus harum duarum solutionum maxime sit absconditus, casum specialem accuratius perpendi expediet. Sit igitur

$$F = 1, \quad G = 0 \text{ et } H = 0,$$

ac binae priores aequationes inter r et s has induent formas:

$$\text{I. } \frac{-dr}{dx} + rr - 2s = A \text{ et}$$

$$\text{II. } \frac{rds}{dx} + ss - rrs + As = B$$

posteriores vero istas:

$$\text{III. } \frac{dp}{dx} - pp + C = 0 \text{ et}$$

$$\text{IV. } \frac{dq}{dx} - qq + \frac{sdp}{dx} + D = 0$$

 quas

quas cum illis certum eſt ita cohaerere vt ſit:

$$r = p + q \quad \text{et} \quad s = \frac{dp}{dx} + pq.$$

Vt ſaltem conſenſum a poſteriori agnoſcamus, ſit $C = 0$ et tertia dat $dx = \frac{dp}{mm+pp}$, unde

$$x = \frac{1}{m} \text{ang. tang.} \frac{p}{m} \quad \text{et} \quad p = m \,\text{tang.}\, mx.$$

Hinc cum ſit

$$\frac{dp}{dx} = mm + pp \quad \text{erit}$$

$$s = mm + pp + pq = mm + pr = m(m + r \,\text{tang.}\, mx),$$

qui valor in I. ſubſtitutus dat

$$-\frac{dr}{dx} + rr - 2mr \,\text{tang.}\, mx - 2mm = \text{A}, \quad \text{ſeu}$$

$$\frac{dr}{dx} = rr - 2mr \,\text{tang.}\, mx - 2mm - \text{A}$$

ſecunda vero ob

$$\frac{ds}{dx} = \frac{m\,dr}{dx} \,\text{tang.}\, mx + \frac{mmr}{\text{cof.}\, mx^2}$$

abit in:

$$\frac{mr\,dr}{dx} \,\text{tang.}\, mx = mr^3 \,\text{tang.}\, mx - 2mmrr \,\text{tang.}\, mx^2$$
$$- m(\text{A} + 2mm) r \,\text{tang.}\, mx - m^4 - \text{A}mm + \text{B}$$

ex quibus dr eliminando fit

$$\text{B} = \text{A}mm + m^4.$$

Pro quarta vero ob

$$q = r - p = r - m \,\text{tang.}\, mx,$$

reſulrefultat:

$$\frac{dr}{dx} = rr - 2mr\,\text{tang.}\,mx - mm - D$$

ita vt fit

$$D = mm + A.$$

Confenfus ergo noftrarum aequationum in hac conftantium relatione confiftit vt ob $mm = -C$ fit

$$D = A - C \text{ et } B = -C(A - C) = -CD.$$

In genere vero etiam eaedem relationes locum habent nam fi III. et IV. in vnam fummam colligantur ob

$$C + D = A \text{ et } p + q = r \text{ erit}$$

$$\frac{Fdr}{dx} + Gr + \frac{rdP}{dx} - Fpp - Fqq - 2H - \frac{dG}{dx} + \frac{2Fdp}{dx} + A = 0$$

cum vero fit $\frac{dp}{dx} = s - pq$, fit

$$\frac{Fdr + rdF - dG}{dx} + Gr - Frr - 2H + 2Fs + A = 0 \text{ feu}$$

$$\frac{d.(G - Fr)}{dx} - (G - Fr)r + 2(H - Fs) = A$$

quae eft ipfa aequatio prima. Porro aequatio III. ob $\frac{dp}{dx} = s - pq$ dat

$$Fs - Fpr + Gp - H + C = 0 \text{ feu } C = H - Fs - p(G - Fr)$$

quarta vero reducitur ad hanc formam:

$$\frac{Fdr}{dx} + Gq + \frac{qdF}{dx} - Fqq - H - \frac{dG}{dx} + Fs - Fpq + \frac{pdF}{dx} + D = 0$$

feu

$$\frac{d.(Fr - G)}{dx} + q(G - Fr) - H + Fs + D = 0$$

hincque

$$D = \frac{d.(G - Fr)}{dx} - q(G - Fr) + H - Fs$$

ex

ex quibus concluditur:

$$CD = \frac{(H-Fs)d.(G-Fr)}{dx} - q(G-Fr)(H-Fs) + (H-Fs)^2$$
$$- \frac{p(G-Fr)d.(G-Fr)}{dx} + pq(G-Fr)^2 - p(G-Fr)(H-Fs).$$

Ex ſecunda vero habemus:

$$B = \frac{(G-Fr)d.(H-Fs)}{dx} - \frac{(H-Fs)d.(G-Fr)}{dx} - (H-Fs)^2$$
$$+ (G-Fr)(H-Fs)r - (G-Fr)^2 s$$

quibus expreſſionibus coniunctis fit

$$\frac{CD+B}{G-Fr} = \frac{d.(H-Fs)}{dx} - \frac{p\,d.(G-Fr)}{dx} - \frac{dp(G-Fr)}{dx}$$
$$= \frac{d.(H-Fs) - d.p(G-Fr)}{dx} = 0$$

ſiquidem eſt

$$C = H - Fs - p(G-Fr),$$

ex quo etiam in genere eſt

$$B = -CD \quad \text{et} \quad A = C + D.$$

Interim tamen hinc non perſpicitur, quomodo ex aeqq. I. et II. binae reliquae III. et IV. deriuari queant.

Scholion 4.

374. Omnibus his diligenter penſitatis manifeſtum fiet totum negotium ope ſubſtitutionis ſatis ſimplicis confici poſſe. Quod quo facilius oſtendatur, ponamus breuitatis cauſa

$$G - Fr = R \quad \text{et} \quad H - Fs = S$$

vt

vt habeantur hae duae aequationes:

$$\text{I. } A = \frac{dR}{dx} - \frac{GR}{F} + \frac{RR}{F} + 2S$$

$$\text{II. } B = \frac{RdS - SdR}{dx} - \frac{HRR}{F} + \frac{GRS}{F} - SS$$

ex quibus duas quantitates R et S erui oporteat, dum F, G, H sunt functiones quaecunque ipsius x, at A et B quantitates constantes. Ad hoc adhibeatur ista substitutio $S = C + Rp$ ita adornanda, vt binae illae aequationes coalescant in vnam, in qua praeter x vnica insit noua variabilis p deinceps per methodos cognitas inuestiganda. Hinc ob

$dS = Rdp + pdR$ habebitur

$$\text{I. } A = \frac{dR}{dx} - \frac{GR}{F} + \frac{RR}{F} + 2C + 2Rp$$

$$\text{II. } B = \frac{RRdp}{dx} - \frac{CdR}{dx} - \frac{HRR}{F} + \frac{CGR}{F} + \frac{GRRp}{F} - CC - 2CRp - RRpp$$

vnde primo eliminando dR concluditur:

$$B + AC = \frac{RRdp}{dx} + \frac{CRR}{F} + CC - \frac{HRR}{F} - RRpp$$

dummodo ergo constantem C ita assumamus, vt sit $CC = B + AC$, per diuisionem etiam ipsa quantitas R tolletur, resultabitque haec aequatio:

$$0 = \frac{dp}{dx} + \frac{C}{F} - \frac{H}{F} - pp$$

cuius resolutio ad methodos, magis cognitas pertinet. Cum igitur ista methodus maximi sit momenti, sequens problema, etiamsi ad primam partem calculi integralis sit referendum, hic adiicere operae pretium videtur.

Problema 60.

375. Propositis huiusmodi duabus aequationibus differentialibus:

$$\text{I. } 0 = \frac{dy}{dx} + F + Gy + Hz + Iyy + Kyz + Lzz$$

$$\text{II. } 0 = \frac{ydz - zdy}{dx} + P + Qy + Rz + Syy + Tyz + Vzz$$

vbi F, G, H etc. P, Q, R etc. sint functiones ipsius x, methodum exponere has aequationes, siquidem fieri licet, resoluendi.

Solutio.

Methodus indicata in hoc consistit, vt ope substitutionis $z = a + yv$ ex illis aequationibus vna elici queat duas tantum variabiles x et v implicans. Quoniam igitur est

$$ydz - zdy = yydv - ady,$$

ex I. a + II. nascitur haec aequatio:

$$0 = \frac{yydv}{dx} + P + Qy + Rz + Syy + Tyz + Vzz$$
$$+ aF + aGy + aHz + aIyy + aKyz + aLzz$$

quae loco z substituto valore $a + yv$ ita exhibeatur, secundum potestates ipsius y

$$0 = \frac{yydv}{dx} + y^0(P + aF + a(R + aH) + aa(V + aL))$$
$$+ y^1(Q + aG + v(R + aH) + a(T + aK) + 2av(V + aL))$$
$$+ y^2(S + aI + v(T + aK) + vv(V + aL))$$

nunc-

nuncque efficiendum eſt, vt tota aequatio per yy diuidi queat, ideoque partes per y^0 et y^1 affectae euaneſcant. Ex parte ergo y^0 fieri oportet:

$$P + aF + a(R + aH) + aa(V + aL) = 0$$

ex parte autem y^1, quia v eſt noua variabilis in calculum inducta hae duae conditiones naſcuntur:

$$Q + aG + a(T + aK) = 0 \text{ et}$$

$$R + aH + 2a(V + aL) = 0$$

vnde prima dabit

$$P + aF - aa(V + aL) = 0.$$

Conditiones ad iſtam reductionem requiſitae ſunt hae tres:

$$\text{I. } P + aF - aa(V + aL) = 0$$

$$\text{II. } Q + aG + a(T + aK) = 0$$

$$\text{III. } R + aH + 2a(V + aL) = 0$$

vnde vel P, Q et R vel F, G et H commode definiuntur.

His autem conditionibus ſtabilitis totum negotium ad reſolutionem huius aequationis reuocatur:

$$0 = \frac{dv}{dx} + S + aI + v(T + aK) + vv(V + aL)$$

quae duas tantum continet variabiles x et v, ex qua v per x determinari oportet. Cum deinde poſito $z = a + yv$ prima aequatio induat hanc

 for-

formam:

$$o=\frac{dy}{dx}+F+aH+aaL+y(G+Hv+aK+2aLv)$$
$$+yy(I+Kv+Lvv)$$

secunda vero istam

$$o=\frac{yydv}{dx}-\frac{ady}{dx}+P+aR+aaV+y(Q+Rv+aT+2aVv)$$
$$+yy(S+Tv+Vvv)$$

seu hinc superiorem per yy multiplicatam subtrahendo:

$$o=\frac{-ady}{dx}+P+aR+aaV+y(Q+Rv+aT+2aVv)$$
$$-yy(Ia+aKv+aLvv)$$

quae quidem cum illa congruit, vt natura rei postulat.

Coroll. 1.

373. Si ergo huiusmodi binae aequationes fuerint propositae

$$o=\frac{dy}{dx}+F+Gy+Hz+Iyy+Kyz+Lzz$$
$$o=\frac{ydz-zdy}{dx}-aF-aGy-aHz+Syy+Tyz+Vzz$$
$$+a^2L-aaKy-2aaLz$$
$$+aaV-aTy-2aVz$$

facto $z=a+yv$ primo resolui debet haec aequatio:

$$o=\frac{dv}{dx}+S+aI+v(T+aK)+vv(V+aL)$$

vnde

vnde definita v per x hanc aequationem tractari oportet:

$$0=\frac{dy}{dx}+F+aH+aaL+y(G+aK)+yy(I+Kv+Lvv)$$
$$+vy(H+2aL)$$

quo facto habebitur quoque $z=a+vy$.

Coroll. 2.

377. Si $F=A$, $K=0$, $L=0$, $H=-2b$, $V=b$ et $T=-G$ casus supra 374. tractatus resultat, harum aequationum:

$$0=\frac{dy}{dx}+A+Gy-2bz+Iyy$$

$$0=\frac{ydz-zdy}{dx}-aA+Syy-Gyz+bzz$$
$$+aab$$

vbi G, I et S sunt functiones quaecunque ipsius x, et resolutio ita se habet, vt posito $z=a+yv$ hae aequationes successiue debeant expediri:

$$0=\frac{dv}{dx}+S+aI-Gv+bvv \text{ et}$$

$$0=\frac{dy}{dx}+A-2ab+y(G-2bv)+Iyy.$$

 Coroll. 3.

Coroll. 3.

378. Euidens eſt poſtremam aequationem nulla laborare difficultate etiam in genere dum ſit

$$F + aH + aaL = 0,$$

prioris autem ſolutio in promtu eſt ſi ſit vel $S + aT = 0$, vel $V + aL = 0$.

CALCVLI

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LIBER POSTERIOR.

PARS PRIMA

SEV

INVESTIGATIO FVNCTIONVM DVARVM VARIABILIVM EX DATA DIFFERENTIALIVM CVIVSVIS GRADVS RELATIONE.

SECTIO TERTIA

INVESTIGATIO DVARVM VARIABILIVM FVNCTIONVM EX DATA DIFFERENTIALIVM TERTII ALTIORVMQVE GRADVVM RELATIONE.

CAPVT I.

DE
RESOLVTIONE AEQVATIONVM SIMPLICISSIMARVM VNICAM FORMVLAM DIFFERENTIALEM INVOLVENTIVM.

Problema 61.

379.

Indolem functionis binarum variabilium x et y indagare, si eius quaepiam formula differentialis tertii gradus euanescat.

Solutio.

Sit z functio illa quaesita, et cum eius sint quatuor formulae differentiales tertii gradus

$$\left(\frac{d^3 z}{d x^3}\right), \left(\frac{d^3 z}{d x^2 d y}\right), \left(\frac{d^3 z}{d x d y^2}\right) \text{ et } \left(\frac{d^3 z}{d y^3}\right),$$

prout quaelibet harum nihilo aequalis ſtatuitur, totidem habemus caſus euoluendos.

I. Sit igitur primo $(\frac{d^3 z}{dx^3})=0$, et ſumta y conſtante prima integratio praebet

$$(\frac{ddz}{dx^2})=\Gamma:y;$$

tum ſimili modo ſecunda integratio dat

$$(\frac{dz}{dx})=x\Gamma:y+\Delta:y,$$

vnde tandem fit

$$z=\tfrac{1}{2}xx\Gamma:y+x\Delta:y+\Sigma:y$$

vbi $\Gamma:y$, $\Delta:y$ et $\Sigma:y$ denotant functiones quascunque ipſius y, ita vt ob triplicem integrationem tres functiones arbitrariae in calculum ſint ingreſſae, vt rei natura poſtulat.

II. Sit $(\frac{d^3 z}{dx^2 dy})=0$, ac primo bis integrando per ſolius x variabilitatem reperitur vt ante:

$$(\frac{dz}{dy})=x\Gamma':y+\Delta':y$$

nunc autem ſola y pro variabili habita, adipiſcimur:

$$z=x\Gamma:y+\Delta:y+\Sigma:x$$

quandoquidem apices ſignis functionum inſcripti hic ſemper hunc habent ſignificatum vt ſit

$$\int dy\Gamma':y=\Gamma:y \text{ et } \int dy\Delta':y=\Delta:y.$$

III. Sit $(\frac{d^3 z}{dx\,dy^2})=0$, et quia hic caſus a praecedente non differt, niſi quod binae variabiles x et y inter

inter se sint permutatae, integrale quaesitum est
$z = y\Gamma : x + \Delta : x + \Sigma : y.$

IV. Sit $\left(\frac{d^3 z}{d y^3}\right) = 0$ et ob similem permutationem ex casu primo intelligitur fore:
$z = \frac{1}{2} y^2 \Gamma : x + y \Delta : x + \Sigma : x.$

Coroll. 1.

380. Tres functiones arbitrariae, hic per triplicem integrationem ingressae sunt vel ipsius x, vel ipsius y tantum; omnes tres sunt ipsius y tantum casu primo $\left(\frac{d^3 z}{d x^3}\right) = 0$, ipsius x vero tantum casu quarto $\left(\frac{d^3 z}{d y^3}\right) = 0$; duae vero sunt ipsius y et vna ipsius x casu secundo $\left(\frac{d^3 z}{d x^2 d y}\right) = 0$; contra autem duae ipsius x et vna ipsius y casu tertio $\left(\frac{d^3 z}{d x d y^2}\right) = 0.$

Coroll. 2.

381. Porro obseruasse iuuabit, si eiusdem variabilis puta x duae pluresue occurrant functiones arbitrariae, vnam quidem absolute poni, alteram per y multiplicari, tertiam vero si adsit per $\frac{1}{2} yy$ seu quod eodem redit per yy multiplicatam accedere.

Coroll. 3.

382. Perpetuo autem tenendum est has functiones ita arbitrio nostro relinqui, vt etiam functiones discontinuae seu nulla continuitatis lege contentae non excludantur. Scilicet si libero manus tractu linea quaecunque describatur, applicata respondens abscissae x huiusmodi functionem $\Gamma : x$ referet.

Scholion 1.

383. Minus hic immorandum arbitror transformationi formularum differentialium altioris gradus, dum loco binarum variabilium x et y aliae quaecunque in calculum introducuntur, quoniam in genere expressiones nimis fierent complicatae vixque vllum vsum habiturae, tum vero imprimis quod methodus has transformationes inueniendi iam supra (229.) satis luculenter est tradita. Casum tantum simpliciorem, quo binae nouae variabiles t et u loco x et y introducendae ita accipiuntur, vt sit

$$t = \alpha x + \beta y \text{ et } u = \gamma x + \delta y$$

hic quoque ad formulas differentiales altiores accommodabo. Cum igitur viderimus esse

pro formulis primi gradus:

$$\left(\frac{dz}{dx}\right) = \alpha\left(\frac{dz}{dt}\right) + \gamma\left(\frac{dz}{du}\right)$$

$$\left(\frac{dz}{dy}\right) = \beta\left(\frac{dz}{dt}\right) + \delta\left(\frac{dz}{du}\right)$$

et pro formulis secundi gradus:

$$\left(\frac{ddz}{dx^2}\right) = \alpha^2\left(\frac{ddz}{dt^2}\right) + 2\alpha\gamma\left(\frac{ddz}{dt\,du}\right) + \gamma^2\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dx\,dy}\right) = \alpha\beta\left(\frac{ddz}{dt^2}\right) + (\alpha\delta + \beta\gamma)\left(\frac{ddz}{dt\,du}\right) + \gamma\delta\left(\frac{ddz}{du^2}\right)$$

$$\left(\frac{ddz}{dy^2}\right) = \beta^2\left(\frac{ddz}{dt^2}\right) + 2\beta\delta\left(\frac{ddz}{dt\,du}\right) + \delta^2\left(\frac{ddz}{du^2}\right)$$

erit

erit pro formulis tertii gradus:

$$\left(\frac{d^3z}{dx^3}\right)=\alpha^3\left(\frac{d^3z}{dt^3}\right)+3\alpha^2\gamma\left(\frac{d^3z}{dt^2du}\right)+3\alpha\gamma^2\left(\frac{d^3z}{dt\,du^2}\right)+\gamma^3\left(\frac{d^3z}{du^3}\right)$$

$$\left(\frac{d^3z}{dx^2dy}\right)=\alpha^2\beta\left(\frac{d^3z}{dt^3}\right)+(\alpha^2\delta+2\alpha\beta\gamma)\left(\frac{d^3z}{dt^2du}\right)+(\beta\gamma^2+2\alpha\gamma\delta)\left(\frac{d^3z}{dt\,du^2}\right)+\gamma^2\delta\left(\frac{d^3z}{du^3}\right)$$

$$\left(\frac{d^3z}{dxdy^2}\right)=\alpha\beta^2\left(\frac{d^3z}{dt^3}\right)+(\beta\beta\gamma+2\alpha\beta\delta)\left(\frac{d^3z}{dt^2du}\right)+(\alpha\delta^2+2\beta\gamma\delta)\left(\frac{d^3z}{dt\,du^2}\right)+\gamma\delta^2\left(\frac{d^3z}{du^3}\right)$$

$$\left(\frac{d^3z}{dy^3}\right)=\beta^3\left(\frac{d^3z}{dt^3}\right)+3\beta^2\delta\left(\frac{d^3z}{dt^2du}\right)+3\beta\delta^2\left(\frac{d^3z}{dt\,du^2}\right)+\delta^3\left(\frac{d^3z}{du^3}\right)$$

et pro formulis quarti gradus:

	$\left(\frac{d^4z}{dt^4}\right)$	$\left(\frac{d^4z}{dt^3du}\right)$	$\left(\frac{d^4z}{dt^2du^2}\right)$	$\left(\frac{d^4z}{dt\,du^3}\right)$	$\left(\frac{d^4z}{du^4}\right)$
$\left(\frac{d^4z}{dx^4}\right)=$	α^4	$+4\alpha^3\gamma$	$+6\alpha^2\gamma^2$	$+4\alpha\gamma^3$	$+\gamma^4$
$\left(\frac{d^4z}{dx^3dy}\right)=$	$\alpha^3\beta$	$\alpha^3\delta+3\alpha^2\beta\gamma$	$3\alpha^2\gamma\delta+3\alpha\beta\gamma^2$	$+3\alpha\gamma^2\delta+\beta\gamma^3$	$+\gamma^3\delta$
$\left(\frac{d^4z}{dx^2dy^2}\right)=$	$\alpha^2\beta^2$	$2\alpha^2\beta\delta+2\alpha\beta^2\gamma$	$\alpha^2\delta^2+4\alpha\beta\gamma\delta+\beta^2\gamma^2$	$2\alpha\gamma\delta^2+2\beta\gamma^2\delta$	$+\gamma^2\delta^2$
$\left(\frac{d^4z}{dxdy^3}\right)=$	$\alpha\beta^3$	$3\alpha\beta^2\delta+\beta^3\gamma$	$3\alpha\beta\delta^2+3\beta^2\gamma\delta$	$\alpha\delta^3+3\beta\gamma\delta^2$	$+\gamma\delta^3$
$\left(\frac{d^4z}{dy^4}\right)=$	β^4	$+4\beta^3\delta$	$+6\beta^2\delta^2$	$+4\beta\delta^3$	$+\delta^4$

vnde ſimul lex pro altioribus gradibus elucet pro formula ſcilicet generali $\left(\frac{d^{m+n}z}{dx^m dy^n}\right)$ hi coefficientes iidem ſunt qui oriuntur ex euolutione huius formae

$$(\alpha+\gamma v)^m(\beta+\delta v)^n,$$

ſiquidem termini ſecundum poteſtates ipſius v diſponantur.

Scholion 2.

384. Haud alienum fore arbitror euolutionem iſtius formulae ex principiis ante ſtabilitis accuratius docere. Sit igitur

$$s=(\alpha+\gamma v)^m(\beta+\delta v)^n$$

ac ponatur:

$$s = A + Bv + Cv^2 + Dv^3 + Ev^4 + Fv^5 + \text{etc.}$$

vbi quidem primo patet esse $A = \alpha^m \beta^n$; pro reliquis vero coefficientibus inueniendis, sumtis differentialibus logarithmorum habebimus:

$$\frac{ds}{s\,dv} = \frac{m\gamma}{\alpha + \gamma v} + \frac{n\delta}{\beta + \delta v} \text{ ideoque}$$

$$\frac{ds}{dv}(\alpha\beta + (\alpha\delta + \beta\gamma)v + \gamma\delta vv) - s(m\beta\gamma + n\alpha\delta + (m+n)\gamma\delta v) = 0$$

vbi si loco s series assumta substituatur, orietur haec aequatio

$$\begin{array}{lllll}
0 = \alpha\beta B & + 2\alpha\beta C v & + 3\alpha\beta D v^2 & + 4\alpha\beta E v^3 & + 5\alpha\beta F v^4 \text{ etc.} \\
 & + \alpha\delta B & + 2\alpha\delta C & + 2\alpha\delta D & + 4\alpha\delta E \\
 & + \beta\gamma B & + 2\beta\gamma C & + 3\beta\gamma D & + 4\beta\gamma E \\
 & & + \gamma\delta B & + 2\gamma\delta C & + 3\gamma\delta D \\
 - m\beta\gamma A & - m\beta\gamma B & - m\beta\gamma C & - m\beta\gamma D & - m\beta\gamma E \\
 - n\alpha\delta A & - n\alpha\delta B & - n\alpha\delta C & - n\alpha\delta D & - n\alpha\delta E \\
 & - (m+n)\gamma\delta A & - (m+n)\gamma\delta B & - (m+n)\gamma\delta C & - (m+n)\gamma\delta D
\end{array}$$

vnde quilibet coefficiens ex praecedentibus ita definitur

$$A = \alpha^m \beta^n$$

$$B = \frac{m\beta\gamma + n\alpha\delta}{\alpha\beta} A$$

$$C = \frac{(m-1)\beta\gamma + (n-1)\alpha\delta}{2\alpha\beta} B + \frac{(m+n)\gamma\delta}{2\alpha\beta} A$$

$$D = \frac{(m-2)\beta\gamma + (n-2)\alpha\delta}{3\alpha\beta} C + \frac{(m+n-1)\gamma\delta}{3\alpha\beta} B$$

$$E = \frac{(m-3)\beta\gamma + (n-3)\alpha\delta}{4\alpha\beta} D + \frac{(m+n-2)\gamma\delta}{4\alpha\beta} C$$

etc.

His

His igitur coefficientibus inuentis, fi ponatur:

$t = \alpha x + \beta y$ et $u = \gamma x + \delta y$.

transformatio formulae differentialis cuiuscunque ita fe habebit, vt fit

$$\left(\frac{d^{m+n}z}{dx^m dy^n}\right) = A\left(\frac{d^{m+n}z}{dt^{m+n}}\right) + B\left(\frac{d^{m+n}z}{dt^{m+n-1}du}\right)$$
$$+ C\left(\frac{d^{m+n}z}{dt^{m+n-2}du^2}\right) + \text{etc.}$$

Problema 62.

385. Indolem functionis binarum variabilium x et y inueftigare, fi eius formula differentialis cuiuscunque gradus euanefcat.

Solutio.

Ex iis quae de formulis differentialibus tertii gradus nihilo aequatis oftendimus in praecedente problemate fatis perfpicuum eft folutionem huius problematis pro formulis differentialibus quarti gradus ita fe habere.

I. Si fit $\left(\frac{d^4 z}{dx^4}\right) = 0$ erit

$z = x^3\Gamma:y + x^2\Delta:y + x\Sigma:y + \Theta:y.$

II. Si fit $\left(\frac{d^4 z}{dx^3 dy}\right) = 0$ erit

$z = x^2\Gamma:y + x\Delta:y + \Sigma:y + \Theta:x.$

III.

III. Si sit $\left(\frac{d^4 z}{dx^2\,dy^2}\right) = 0$ erit

$$z = x\Gamma:y + \Delta:y + y\Sigma:x + \Theta:x.$$

IV. Si sit $\left(\frac{d^4 z}{dx\,dy^3}\right) = 0$ erit

$$z = \Gamma:y + y^2\Delta:x + y\Sigma:x + \Theta:x.$$

V. Si sit $\left(\frac{d^4 z}{dy^4}\right) = 0$ erit

$$z = y^3\Gamma:x + y^2\Delta:x + y\Sigma:x + \Theta:x$$

vnde simul progressus ad altiores gradus est manifestus.

Coroll. 1.

386. Cum hic quatuor functiones arbitrariae occurrant totidem scilicet, quot integrationes instituti oportet, in hoc ipso criterium integrationis completae continetur.

Coroll. 2.

387. Quin etiam vicissim facile ostenditur formas inuentas aequationi propositae satisfacere. Sic cum pro casu tertio inuenerimus:

$$z = x\Gamma:y + \Delta:y + y\Sigma:x + \Theta:x$$

differentiando hinc colligimus:

Primo $\left(\frac{dz}{dx}\right) = \Gamma:y + y\Sigma':x + \Theta':x$

deinde $\left(\frac{ddz}{dx^2}\right) = y\Sigma'':x + \Theta'':x$

tertio $\left(\frac{d^3 z}{dx^2\,dy}\right) = \Sigma'':x$ et

quarto $\left(\frac{d^4 z}{dx^2\,dy^2}\right) = 0$

eodem-

eodemque peruenitur, quocunque ordine differentiationes vel solam x vel solam y variabilem sumendo, instituantur.

Scholion 1.

388. Hactenus vnam formulam differentialem nihilo esse aequalem assumsimus, calculus autem perinde succedit, si huiusmodi formula functioni cuicunque ipsarum x et y aequalis statuatur: quemadmodum in sequentibus problematibus sum ostensurus. Hoc tantum inculcandum censeo si V fuerit functio quaecunque binarum variabilium x et y, tum $\int V dx$ id denotare integrale, quod obtinetur si sola x pro variabili habeatur, in hac vero formula $\int V dy$ solam y pro variabili haberi: quod idem tenendum est de integrationibus repetitis veluti $\int dx \int V dx$, vbi in vtraque sola x variabilis assumitur, in hac vero $\int dy \int V dx$, postquam integrale $\int V dx$ ex sola ipsius x variabilitate fuerit erutum, tum in altera integratione $\int dy \int V dx$ solam y variabilem accipiendam esse. Et cum perinde vtra integratio prior instituatur, etiam hoc discrimen e modo signandi tolli potest hocque integrale geminatum ita $\int\int V dx dy$ exhiberi: hincque intelligitur, quomodo has formulas:

$$\int\int\int V dx^2 dy \text{ seu } \int^3 V dx^2 dy \text{ et } \int^{m+n} V dx^m dy^n$$

interpretari oporteat; hic scilicet signo integrationis $\int$ indices suffigimus, prorsus vti signo differentiationis d suffigi solent, quippe qui indicant, quoties integratio sit repetenda.

Scholion 2.

389. Singulas has integrationes repetendas ita institui hic assumimus, vt nulla relatio inter binas variabiles x et y in subsidium vocetur, quae circumstantia eo diligentius est animaduertenda, cum vulgo, vbi talibus integrationibus opus est, calculus prorsus diuerso modo institui debeat. Quodsi enim proposito quopiam corpore geometrico eius soliditas seu superficies sit inuestiganda per duplicem integrationem huiusmodi formula $\int\int V\,dx\,dy$ euolui debet, existente V certa functione ipsarum x et y; vbi quidem primo quaeritur integrale $\int V\,dy$ spectata x vt constante; at absoluta integratione ad terminos integrationi praescriptos respici oportet, dum scilicet altero praescribitur, vt hoc integrale $\int V\,dy$ euanescat posito $y = 0$, altero vero id eo vsque extendendum est, donec y datae cuipiam functioni ipsius x aequetur. Tum vero postquam hoc integrale $\int V\,dy$ isto modo fuerit determinatum, altera demum integratio formulae $dx\int V\,dy$ suscipitur, in qua quantitas y non amplius inest, dum eius loco certa quaepiam functio ipsius x est substituta, eaque formula iam reuera vnicam variabilem x complectitur. Hic ergo prima integratione absoluta variabilis y in functionem ipsius x abire est censenda, quam propterea in altera integratione, vbi x est varia-

variabilis, minime vt constantem spectare licebit. Ex quo patet hunc casum toto coelo esse diuersum ab iis integrationibus repetendis, quas hic contemplamur, ad quem propterea hic eo minus respicimus, cum ista peculiaris ratio tantum in formula $\iint V\,dx\,dy$ locum habere possit; reliquis vero vbi alterum differentiale dx vel dy saepius repetitur, adeo aduersetur. Quam ob causam hinc omnem relationem, quae forte peracta vna integratione inter binas variabiles x et y statui posset, merito remouemus.

Problema 63.

390. Si formula quaepiam differentialis tertii altiorisue gradus aequetur functioni cuicunque binarum variabilium x et y, indolem functionis z definire.

Solutio.

Sit V functio quaecunque binarum variabilium x et y et incipientes a formulis tertii ordinis sit primo $\left(\frac{d^3 z}{dx^3}\right) = V$, et posita sola x variabili erit

$$\left(\frac{ddz}{dx^2}\right) = \int V\,dx + \Gamma : y\,;$$

tum vero porro

$$\left(\frac{dz}{dx}\right) = \int dx \int V\,dx + x\Gamma : y + \Delta : y = \iint V\,dx^2 + x\Gamma : y + \Delta : y$$

ac denique

$$z = \int^3 V\,dx^3 + \tfrac{1}{2}x^2\Gamma : y + x\Delta : y + \Sigma : y$$

 simili

simili modo patet si fuerit $\left(\frac{d^3 z}{dx^2 dy}\right) = V$ fore

$$z = \int^3 V dx^2 dy + x\Gamma : y + \Delta : y + \Sigma : x$$

ac si sit $\left(\frac{d^3 z}{dx dy^2}\right) = V$ erit

$$z = \int^3 V dx dy^2 + \Gamma : y + y\Delta : x + \Sigma : x$$

si sit $\left(\frac{d^3 z}{dy^3}\right) = V$ erit

$$z = \int^3 V dy^3 + y^2\Gamma : x + y\Delta : x + \Sigma : x.$$

Eodem modo ad formulas altiorum graduum progredientes reperiemus vt sequitur:

si sit $\left(\frac{d^4 z}{dx^4}\right) = V$ fore

$$z = \int^4 V dx^4 + x^3\Gamma : y + x^2\Delta : y + x\Sigma : y + \Theta : y$$

si sit $\left(\frac{d^4 z}{dx^3 dy}\right) = V$ fore

$$z = \int^4 V dx^3 dy + x^2\Gamma : y + x\Delta : y + \Sigma : y + \Theta : x$$

si sit $\left(\frac{d^4 z}{dx^2 dy^2}\right) = V$ fore

$$z = \int^4 V dx^2 dy^2 + x\Gamma : y + \Delta : y + y\Sigma : x + \Theta : x$$

si sit $\left(\frac{d^4 z}{dx dy^3}\right) = V$ fore

$$z = \int^4 V dx dy^3 + \Gamma : y + y^2\Delta : x + y\Sigma : x + \Theta : x$$

si sit $\left(\frac{d^4 z}{dy^4}\right) = V$ fore

$$z = \int^4 V dy^4 + y^3\Gamma : x + y^2\Delta : x + y\Sigma : x + \Theta : x$$

neque pro altioribus gradibus res eget vlteriori explicatione.

Coroll. 1.

Coroll. 1.

391. Quemadmodum ſignum integrationis in primo libro vſitatum iam per ſe inuoluit conſtantem per integrationem ingredientem ita quoque hic functiones arbitrariae per integrationem ingreſſae iam in formula integrali inuolui ſunt cenſendae, ita vt non ſit opus eas exprimere.

Coroll. 2.

392. Sufficit ergo pro aequatione $(\frac{d^3 z}{d x^3}) = V$ integrale triplicatum hoc modo dediſſe $z = \int^3 V d x^3$, quae forma iam poteſtate complectitur partes ſupra adiectas:

$$x x \Gamma : y + x \Delta : y + \Sigma : y :$$

quod idem de reliquis eſt tenendum.

Coroll. 3.

393. Si ergo in genere haec habeatur aequatio

$$\left(\frac{d^{m+n} z}{d x^m d y^n}\right) = V$$

eius integrale ſtatim hoc modo exhibetur:

$$z = \int^{m+n} V d x^m d y^n$$

quae poteſtate iam inuoluit omnes illas functiones arbitrarias numero $m+n$ per totidem integrationes inuectas.

 Scholion.

Scholion.

394. Hi casus vtique sunt simplicissimi, qui ad hoc caput referendi videntur, pro magis autem complicatis vix certa praecepta tradere licet, cum ista calculi integralis pars vix adhuc coli sit coepta. Interim tamen iam intelligitur, si aequationes magis complicatae ope cuiusdam transformationis ad has simplicissimas reuocare liceat, etiam earum integrationem in promtu esse futuram; quod quidem negotium hic non copiosius persequendum videtur. Progredior igitur ad casus magis reconditos, eosque ita comparatos, vt ope aequationum inferiorum ordinum expediri queant, vnde quidem insignis methodus satis late patens colligi poterit, qua saepius haud sine successu vti licebit. Neque tamen in hac pertractatione nimis diffusum esse conuenit, sed sufficiet praecipuos fontes adhuc quidem cognitos patefecisse.

CAPVT II.

CAPVT II.

DE INTEGRATIONE AEQVATIONVM ALTIORVM PER REDVCTIONEM AD INFERIORES.

Problema 64.

395.

Propoſita hac aequatione tertii gradus $\left(\frac{d^3 z}{d x^3}\right) = a^3 z$ indolem functionis z inueſtigare.

Solutio.

Fingatur huic aequationi ſatisfacere haec ſimplicior primi gradus

$$\left(\frac{d z}{d x}\right) = n z,$$

et cum hinc differentiando obtineatur

$$\left(\frac{d d z}{d x^2}\right) = n \left(\frac{d z}{d x}\right) = n n z$$

hincque porro

$$\left(\frac{d^3 z}{d x^3}\right) = n n \left(\frac{d z}{d x}\right) = n^3 z,$$

euidens eſt quaeſito ſatisfieri, dum ſit $n^3 = a^3$, id quod

quod triplici modo euenire potest:

I. $n = a$;

II. $n = \frac{-1+\sqrt{-3}}{2} a$;

III. $n = \frac{-1-\sqrt{-3}}{2} a$.

Pro quolibet ergo valore quaeratur integrale completum aequationis $\left(\frac{dz}{dx}\right) = nz$, et tria haec integralia coniuncta praebebunt integrale completum aequationis propositae. Cum autem in aequatione $\left(\frac{dz}{dx}\right) = nz$ quantitas y constans sumatur, erit

$$dz = nz\,dx \text{ seu } \frac{dz}{z} = n\,dx$$

vnde fit

$$lz = nx + l\Gamma:y \text{ seu } z = e^{nx}\Gamma:y.$$

Tribuantur iam ipsi n terni valores, eritque pro aequatione proposita:

$$z = e^{ax}\Gamma:y + e^{\frac{-1+\sqrt{-3}}{2}ax}\Delta:y + e^{\frac{-1-\sqrt{-3}}{2}ax}\Sigma:y.$$

Cum autem sit

$$e^{m\sqrt{-1}} = \operatorname{cos}.m + \sqrt{-1}.\operatorname{sin}.m,$$

erit functionum arbitrariarum formam mutando:

$$z = e^{ax}\Gamma:y + e^{-\frac{1}{2}ax}\operatorname{cos}.\tfrac{ax\sqrt{3}}{2}.\Delta:y + e^{-\frac{1}{2}ax}\operatorname{sin}.\tfrac{ax\sqrt{3}}{2}.\Sigma:y.$$

Coroll. 1.

Coroll. 1.

396. Integrale hoc etiam ita repraesentari potest:

$$z = e^{ax}\Gamma : y + e^{-\frac{1}{2}ax}\Delta : y . \cos\left(\tfrac{ax\sqrt{3}}{2} + Y\right)$$

denotante Y functionem quamcunque ipsius y.

Coroll. 2.

397. Quia tribus integrationibus est opus, et in singulis quantitas y vt constans tractatur; secundum praecepta libri primi haec aequatio $d^3z = a^3zdx^3$ resoluatur, et loco trium constantium functiones quaecunque ipsius y introducantur; vnde eadem solutio elicitur.

Problema 65.

398. Proposita hac aequatione cuiuscunque gradus:

$$Pz + Q\left(\tfrac{dz}{dx}\right) + R\left(\tfrac{ddz}{dx^2}\right) + S\left(\tfrac{d^3z}{dx^3}\right) + T\left(\tfrac{d^4z}{dx^4}\right) \text{ etc.} = 0$$

vbi litterae P, Q, R, S, T etc. functiones denotant quascunque binarum variabilium x et y, indolem functionis z definire.

Solutio.

Cum in omnibus integrationibus instituendis quantitas y perpetuo vt constans spectetur, haec aequatio inter duas tantum variabiles x et z consistere est censenda. Quare per praecepta libri primi haec tractanda erit aequatio:

$$Pz + \frac{Q\,dz}{dx} + \frac{R\,ddz}{dx^2} + \frac{S\,d^3z}{dx^3} + \frac{T\,d^4z}{dx^4} + \text{etc.} = 0$$

cuius resolutio si succedat, tantum opus est, vt loco constantium per singulas integrationes inuectarum functiones quaecunque ipsius y scribantur; sicque habebitur integrale desideratum, idque completum siquidem hanc aequationem complete integrare licuerit.

Coroll. 1.

399. Si ergo litterae P, Q, R, S etc. sint constantes vel solam variabilem y inuoluant, integratio semper succedit, quoniam in primo libro huiusmodi aequationes in genere integrare docuimus.

Coroll. 2.

400. Deinde etiam resolutio succedit huius aequationis

$$Az + Bx\left(\frac{dz}{dx}\right) + Cx^2\left(\frac{ddz}{dx^2}\right) + Dx^3\left(\frac{d^3z}{dx^3}\right) + \text{etc.} = 0$$

siue litterae A, B, C etc. sint constantes siue functiones ipsius y tantum.

Coroll. 3.

401. Tum vero etiam si hae formae non sint aequales nihilo, sed functioni cuicunque ipsarum

rum x et y aequentur resolutio nihilo minus succedit, per ea, quae in postremis capitibus libri primi sunt exposita.

Scholion.

402. Haec etiam multo latius extendi possunt ad omnes plane aequationes, in quibus nullae aliae formulae differentiales praeter has

$$\left(\frac{dz}{dx}\right), \left(\frac{ddz}{dx^2}\right), \left(\frac{d^3z}{dx^3}\right) \text{ etc.}$$

quae solam x vt variabilem implicant occurrunt. Quomodocunque enim istae formulae cum quantitatibus finitis x, y et z fuerint complicatae, aequatio semper ad librum primum pertinere est censenda; quoniam in omnibus integrationibus instituendis quantitas y perpetuo vt constans tractatur. Confectis demum integrationibus dicrimen in hoc consistit, vt loco constantium arbitrariarum functiones arbitrariae ipsius y in calculum introducantur. Superfluum foret hic monere, quae de altera variabilium y sunt dicta, etiam de altera x esse intelligenda.

Problema 66.

403. Proposita hac aequatione

$$\left(\frac{ddz}{dx^2}\right) + b\left(\frac{ddz}{dx\,dy}\right) - 2a\left(\frac{dz}{dx}\right) - ab\left(\frac{dz}{dy}\right) + aaz = 0$$

inuestigare indolem functionis z.

Solutio.

Facile patet huic aequationi satisfacere hanc aequationem simplicem $\left(\frac{dz}{dx}\right)=az$, vnde fit $z=e^{ax}$; statuamus ergo $z=e^{ax}v$ eritque:

$$\left(\frac{dz}{dx}\right)=e^{ax}\left(av+\left(\frac{dv}{dx}\right)\right),\ \left(\frac{dz}{dy}\right)=e^{ax}\left(\frac{dv}{dy}\right)$$

hincque

$$\left(\frac{ddz}{dx^2}\right)=e^{ax}\left(aav+2a\left(\frac{dv}{dx}\right)+\left(\frac{ddv}{dx^2}\right)\right) \text{ et}$$

$$\left(\frac{ddz}{dxdy}\right)=e^{ax}\left(a\left(\frac{dv}{dy}\right)+\left(\frac{ddv}{dxdy}\right)\right)$$

quibus valoribus substitutis et diuisa aequatione per e^{ax} habebimus:

$$\left(\frac{ddv}{dx^2}\right)+b\left(\frac{ddv}{dxdy}\right)=0.$$

Quia nunc hic vbique occurrit $\left(\frac{dv}{dx}\right)$ faciamus $\left(\frac{dv}{dx}\right)=u$ erit

$$\left(\frac{du}{dx}\right)+b\left(\frac{du}{dy}\right)=0;$$

cuius integrale est

$$f:(y-bx)=u$$

scribamus ergo

$$u=\left(\frac{dv}{dx}\right)=-b\Gamma:(y-bx),$$

vt prodeat

$$v=\Gamma:(y-bx)+\Delta:y,$$

ideoque integrale quaesitum erit:

$$z=e^{ax}(\Gamma:(y-bx)+\Delta:y)$$

quae

quae forma ob duas functiones arbitrarias vtique est integrale completum.

Problema 67.

404. Proposita hac aequatione:

$$0=(a+2b)z-(2a+3b)\left(\frac{dz}{dx}\right)+c\left(\frac{dz}{dy}\right)+a\left(\frac{ddz}{dx^2}\right)-2c\left(\frac{ddz}{dxdy}\right)$$
$$+b\left(\frac{d^3z}{dx^3}\right)+c\left(\frac{d^3z}{dx^2dy}\right)$$

indolem functionis z inuestigare.

Solutio.

Aequatio haec ita est comparata vt ei manifesto satisfaciat $z=e^x$, statuamus ergo $z=e^x v$ eritque

$$\left(\frac{dz}{dx}\right)=e^x\left(v+\left(\frac{dv}{dx}\right)\right);\ \left(\frac{dz}{dy}\right)=e^x\left(\frac{dv}{dy}\right)$$
$$\left(\frac{ddz}{dx^2}\right)=e^x\left(v+2\left(\frac{dv}{dx}\right)+\left(\frac{ddv}{dx^2}\right)\right);\ \left(\frac{ddz}{dxdy}\right)=e^x\left(\left(\frac{dv}{dy}\right)+\left(\frac{ddv}{dxdy}\right)\right)$$
$$\left(\frac{d^3z}{dx^3}\right)=e^x\left(v+3\left(\frac{dv}{dx}\right)+3\left(\frac{ddv}{dx^2}\right)+\left(\frac{d^3v}{dx^3}\right)\right)$$
$$\left(\frac{d^3z}{dx^2dy}\right)=e^x\left(\left(\frac{dv}{dy}\right)+2\left(\frac{ddv}{dxdy}\right)+\left(\frac{d^3v}{dx^2dy}\right)\right)$$

quibus valoribus substitutis emergit haec satis simplex aequatio:

$$0=(a+3b)\left(\frac{ddv}{dx^2}\right)+b\left(\frac{d^3v}{dx^3}\right)+c\left(\frac{d^3v}{dx^2dy}\right)$$

in qua commode euenit vt in singulis terminis formula $\left(\frac{ddv}{dx^2}\right)$ contineatur, quare posito $\left(\frac{ddv}{dx^2}\right)=u$, prodit haec aequatio primi gradus

$$0=(a+3b)u+b\left(\frac{du}{dx}\right)+c\left(\frac{du}{dy}\right)$$

ex qua patet si ponatur

$$du = p\,dx + q\,dy$$

esse debere

$$(a+3b)u + bp + cq = 0$$

quae ita resoluitur.

Cum posito $a+3b=f$ fit

$$q = -\frac{bp}{c} - \frac{fu}{c} \text{ erit}$$

$$du = p\,dx - \frac{bp\,dy}{c} - \frac{fu\,dy}{c} \text{ seu}$$

$$dx - \frac{b\,dy}{c} = \frac{1}{p}\left(du + \frac{fu\,dy}{c}\right) = \frac{u}{p}\left(\frac{du}{u} + \frac{f\,dy}{c}\right)$$

sicque necesse est vt sit $\frac{u}{p}$ functio ipsius $x - \frac{by}{c}$, vnde fit

$$lu + \frac{fy}{c} = f:(cx - by) \text{ et}$$

$$u = e^{\frac{-fy}{c}}\,\Gamma''\!:\left(x - \frac{by}{c}\right) = \left(\frac{ddv}{dx^2}\right).$$

Iam ob y constans spectandum prima integratio dat

$$\left(\frac{dv}{dx}\right) = e^{\frac{-fy}{c}}\,\Gamma'\!:\left(x - \frac{by}{c}\right) + \Delta:y$$

et altera

$$v = e^{\frac{-fy}{c}}\,\Gamma:\left(x - \frac{by}{c}\right) + x\,\Delta:y + \Sigma:y.$$

Quare posito $a+3b=f$ aequationis propositae integrale completum est

$$z = e^{x - \frac{fy}{c}}\,\Gamma:\left(x - \frac{by}{c}\right) + e^x x\,\Delta:y + e^x\,\Sigma:y.$$

Problema

Problema 63.

405. Propofita hac aequatione differentiali tertii gradus

$$0 = Pz - 3P\left(\frac{dz}{dx}\right) + 3P\left(\frac{ddz}{dx^2}\right) - P\left(\frac{d^3z}{dx^3}\right) + Q\left(\frac{dz}{dy}\right)$$
$$- 2Q\left(\frac{ddz}{dx\,dy}\right) + Q\left(\frac{d^3z}{dx^2\,dy}\right)$$

vbi P et Q fint functiones quaecunque ipfarum x et y inueftigare indolem functionis z.

Solutio.

Facta fubftitutione $z = e^x v$, quandoquidem ex data forma facile perfpicitur valorem e^x loco z pofitum fatisfacere, peruenitur ad hanc aequationem:

$$-P\left(\frac{d^3v}{dx^3}\right) + Q\left(\frac{d^3v}{dx^2\,dy}\right) = 0$$

quae porro pofito $\left(\frac{ddv}{dx^2}\right) = u$ vt fit $v = \int\int u\,dx^2$ abit in hanc

$$-P\left(\frac{du}{dx}\right) + Q\left(\frac{du}{dy}\right) = 0.$$

Statuamus $du = p\,dx + q\,dy$ erit $Qq = Pp$, hinc $q = \frac{Pp}{Q}$ ideoque

$$du = p\left(dx + \frac{P}{Q}dy\right);$$

ex quo intelligitur quantitatem p ita comparatam effe debere, vt formula

$$dx + \frac{P}{Q}dy$$

per eam multiplicata integrabilis euadat. Quaeratur ergo multiplicator M formulam

$$Q\,dx + P\,dy$$

inte-

integrabilem reddens ita vt fit

$$\int M(Q\,dx + P\,dy) = s,$$

quam ergo functionem s ipsarum x et y inueniri posse assumo et ob

$$Q\,dx + P\,dy = \frac{ds}{M}$$

habebimus $du = \frac{P\,ds}{MQ}$, vnde patet $\frac{P}{MQ}$ functionem denotare quantitatis s. Posito ergo $\frac{P}{MQ} = \Gamma' : s$, statim erit $u = \Gamma : s$, hincque $v = \int dx \int dx\,\Gamma : s$ in qua vtraque integratione quantitas y vt constans spectatur. Quocirca resolutio problematis ita se habebit:

Pro formula differentiali $Q\,dx + P\,dy$ quaeratur multiplicator M eam reddens integrabilem, vt fit

$$M(Q\,dx + P\,dy) = ds,$$

et inuenta hac ipsarum x et y functione s erit

$$z = e^x \int dx \int dx\,\Gamma : s + e^x x\,\Delta : y + e^x\,\Sigma : y.$$

Scholion.

406. In istis aequationibus hoc commodi vsu venit, vt facta substitutione $z = e^x v$ eiusmodi induant formam, quae facile porro ad speciem simplicem in prima sectione consideratam reuocari queat, etiamsi enim differentialia tertii gradus non sint destructa, tamen reliqua membra ista e calculo excesserunt, vt deinceps noua substitutione $\left(\frac{ddv}{dx^2}\right) = u$ vti, eiusque ope ad aequationem differentialem primi gradus

gradus peruenire licuerit. Vnica igitur subſtitutio hoc praeſtitura fuiſſet, ſi ſtatim poſuiſſemus $z = e^x \int u dx^2$. Vtinam praecepta haberentur, quorum ope huiusmodi ſubſtitutiones facile dignoſci poſſent! Interim poſtremo problemate multo latius patente in ſubſidium vocato §. 209. reſolui poterit.

Problema 67.

407. Propoſita hac aequatione differentiali tertii gradus:

$$0 = (P+Q)z - (2P+3Q)\left(\frac{dz}{dx}\right) + (P+3Q)\left(\frac{ddz}{dx^2}\right) - Q\left(\frac{d^3z}{dx^3}\right)$$
$$-R\left(\frac{dz}{dy}\right) \quad + 2R\left(\frac{ddz}{dx\,dy}\right) \quad - R\left(\frac{d^3z}{dx^2\,dy}\right)$$

vbi P, Q et R ſint functiones quaecunque datae ipſarum x et y, inueſtigare indolem functionis z.

Solutio.

Eadem adhibita ſubſtitutione $z = e^x v$, qua hactenus ſumus vſi, aequatio propoſita transmutatur in ſequentem:

$$0 = P\left(\frac{ddv}{dx^2}\right) - Q\left(\frac{d^3v}{dx^3}\right) - R\left(\frac{d^3v}{dx^2\,dy}\right)$$

vbi commode euenit, vt poſito $\left(\frac{ddv}{dx^2}\right) = u$ iſta reſultet aequatio differentialis primi gradus

$$0 = Pu - Q\left(\frac{du}{dx}\right) - R\left(\frac{du}{dy}\right)$$

vnde qualis ipſarum x et y functio ſit u eſt inquirendum. Ponamus eſſe

$$du = p\,dx + q\,dy,$$

et quia iam illa conditio praebet

$$Pu = Qp + Rq,$$

 ſecun-

ſecundum artificium ſupra §. 209. vſurpatum formemus hinc tres ſequentes aequationes:

$$L\,du = L\,p\,dx + L\,q\,dy$$
$$MP\,u\,dx = MQ\,p\,dx + MR\,q\,dx$$
$$NP\,u\,dy = NQ\,p\,dy + NR\,q\,dy$$

quae in vnam ſummam collectae dabunt:

$$L\,du + Pu(M\,dx + N\,dy) = p((L+MQ)dx + NQ\,dy) + q((L+NR)dy + MR\,dx)$$

vbi cum tres quantitates L, M et N ab arbitrio noſtro pendeant, inter eas ſtatuatur primo eiusmodi relatio, vt binae partes poſterioris membri communem obtineant factorem, ſit ſcilicet:

$$L+MQ : NQ = MR : L+NR \text{ ſeu } L = -MQ - NR$$

et habebimus:

$$-du(MQ+NR) + Pu(M\,dx + N\,dy) = (Mq - Np)(R\,dx - Q\,dy).$$

Quaeratur multiplicator T formulam $R\,dx - Q\,dy$ reddens integrabilem, vt ſit

$$T(R\,dx - Q\,dy) = ds,$$

ex quo tam functio T quam s vt cognita ſpectari poterit et quia nunc habemus:

$$-du(MQ+NR) + Pu(M\,dx + N\,dy) = (Mq - Np)\frac{ds}{T} \text{ ſeu}$$

$$\frac{du}{u} - \frac{P(M\,dx + N\,dy)}{MQ+NR} = \frac{Np - Mq}{u(MQ+NR)} \cdot \frac{ds}{T}.$$

Nunc cum P, Q, R ſint functiones datae ipſarum x et y, probe notandum eſt inter binas nondum defi-

definitas M et N ſemper eiusmodi relationem ſtatui poſſe vt formula $\frac{P(M\,dx+N\,dy)}{MQ+NR}$ integrationem admittat; ſit ergo eius integrale $= lw$, ita vt ſit

$$M\,dx+N\,dy=\frac{MQ+NR}{P}.\frac{dw}{w}, \text{ et}$$

$$\frac{du}{u}=\frac{dw}{w}+\frac{Np-Mq}{Tu(MQ+NR)}.ds.$$

Neceſſe ergo eſt quantitates p et q ita ſint comparatae vt fiat

$$\frac{Np-Mq}{Tu(MQ+NR)}=f':s,$$

hincque

$$lu=lw+f:s.$$

Loco $f:s$ ſcribamus $l\Gamma:s$, vt prodeat:

$$u=w\Gamma:s$$

ac propterea

$$v=\int dx\int w\,dx\,\Gamma:s+x\Delta:y+\Sigma:y.$$

Conſequenter

$$z=e^x\int dx\int w\,dx\,\Gamma:s+e^x x\Delta:y+e^x\Sigma:y.$$

Coroll. 1.

408. Ad hanc ergo ſolutionem ex forma propoſita ſtatim eruendam, primo quaeratur eiusmodi functio ipſarum x et y, quae vocetur s, vt ſit

$$ds=T(R\,dx-Q\,dy)$$

 id

id quod expedietur multiplicatorem T inueſtigando, quo formula differentialis $R\,dx - Q\,dy$ integrabilis reddatur.

Coroll. 2.

409. Praeterea vero quoque quantitatem w inveſtigari oportet. In hunc finem inter quantitates M et N eiusmodi rationem indagari conuenit vt fiat

$$\int \frac{P(M\,dx + N\,dy)}{MQ + NR} = l\,w$$

quae quidem inueſtigatio ſemper eſt concedenda.

Scholion.

410. Cum ſtatim totum negotium eo ſit perductum vt functio u ex hac aequatione definiri debeat

$$Pu = Q\left(\frac{du}{dx}\right) + R\left(\frac{du}{dy}\right)$$

ſine ambagibus, quibus in ſolutione ſum vſus, ſolutio ſequenti modo multo facilius abſolui poterit, id quod inſigne ſupplementum in ſectionem primam ſuppeditat. Statuatur

$$\left(\frac{du}{dx}\right) = LM\,u \text{ et } \left(\frac{du}{dy}\right) = LNu$$

erit primo

$$P = L(MQ + NR), \text{ hinc}$$

$$L = \frac{P}{MQ + NR} \text{ deinde ob}$$

$$du = dx\left(\frac{du}{dx}\right) + dy\left(\frac{du}{dy}\right) \text{ habebimus}$$

$$\frac{du}{u} = L(M\,dx + N\,dy) = \frac{P(M\,dx + N\,dy)}{MQ + NR}$$ vbi

vbi M et N ita accipi oportet, vt integratio succedat, quod cum innumeris modis fieri possit, solutio hinc completa obtineri est aestimanda. Verum dum casus integrationis particularis constet, multo commodius inde solutio completa sequenti ratione elicietur. Posito scilicet

$$\frac{dw}{w} = \frac{P(Mdx+Ndy)}{MQ+NR},$$

ita vt valor ipsius w pro u sumtus iam particulariter satisfaciat, sitque

$$Pw = Q(\tfrac{dw}{dy}) + R(\tfrac{dw}{dy}).$$

Statuamus pro valore completo $u = w\Gamma : s$, et facta substitutione consequimur:

$$Pw\Gamma:s = Q(\tfrac{dw}{dx})\Gamma:s + R(\tfrac{dw}{dy})\Gamma:s$$
$$+ Qw(\tfrac{ds}{dx})\Gamma':s + Rw(\tfrac{ds}{dy})\Gamma':s$$

quae aequatio subito in hanc contrahitur:

$$Q(\tfrac{ds}{dx}) + R(\tfrac{ds}{dy}) = 0,$$

ex qua concludimus

$$(\tfrac{ds}{dx}) = TR \text{ et } (\tfrac{ds}{dy}) = -TQ$$

ac propterea

$$ds = T(Rdx - Qdy),$$

vnde patet hanc quantitatem s inueniri ex formula $Rdx - Qdy$ pro qua primo factor T eam reddens, integrabilem quaeri, tum vero eius integrale pro s sumi debet. Imprimis igitur hic attendatur, quam con-

concinne eandem solutionem elicere liceat, ad quam per tantas ambages perueneramus.

Problema 68.

411. Proposita hac aequatione differentiali quarti gradus:

$$\left(\frac{d^4 z}{d x^4}\right) = aa\left(\frac{d d z}{d x^2}\right).$$

functionis z inuentionem saltem ad resolutionem aequationis simplicioris reducere.

Solutio.

Hanc aequationem attentius contemplanti mox patebit ei satisfacere huiusmodi simpliciorem

$$\left(\frac{d d z}{d y^2}\right) = b\left(\frac{d z}{d x}\right)$$

hinc enim per y differentiando fit

$$\left(\frac{d^3 z}{d y^3}\right) = b\left(\frac{d d z}{d x d y}\right)$$

ac denuo eodem modo

$$\left(\frac{d^4 z}{d y^4}\right) = b\left(\frac{d^3 z}{d x d y^2}\right)$$

at ex ipsa assumta per x differentiata prodit

$$\left(\frac{d^3 z}{d x d y^2}\right) = b\left(\frac{d d z}{d x^2}\right)$$

quo valore ibi inducto colligitur

$$\left(\frac{d^4 z}{d y^4}\right) = bb\left(\frac{d d z}{d x^2}\right),$$

quae forma cum proposita congruit, dum sit $bb = aa$, quod

quod cum duplici modo euenire queat

$b = +a$ et $b = -a$,

poſtquam has aequationes ſimpliciores reſoluerimus:

$\left(\frac{ddz}{dy^2}\right) - a\left(\frac{dz}{dx}\right) = 0$ quae praebeat $z = P$

$\left(\frac{ddz}{dy^2}\right) + a\left(\frac{dz}{dx}\right) = 0$ quae praebeat $z = Q$

erit pro aequatione propoſita:

$z = P + Q$,

et quia tam P quam Q binas functiones arbitrarias inuoluit integrale hoc modo inuentum quatuor eiusmodi functiones complectetur, ideoque erit completum.

Coroll. 1.

412. Solutiones particulares infinitae facile eliciuntur ponendo

$z = e^{\mu x + \nu y}$,

facta enim ſubſtitutione fieri neceſſe eſt

$\nu^4 = \mu\mu aa$ et $\mu = \pm \frac{\nu\nu}{a}$.

Sit $\nu = \lambda a$ erit $\mu = \pm \lambda\lambda a$ et integrale ſatisfaciens

$z = e^{\lambda a (y \pm \lambda x)}$.

Coroll. 2.

413. Poni etiam poteſt

$z = e^{\mu x} \cos.(\nu y + \alpha)$,

vnde

vnde fit

$$\nu^4 = \mu\,\mu\,a\,a$$

vt ante, ita vt alia forma integralium particularium fit

$$z = e^{\pm\lambda\lambda a x}\operatorname{cof.}(\lambda a y + \alpha).$$

Huiusmodi formulae infinitae coniunctae integrale completum quasi exhaurire sunt putandae.

Coroll. 3.

414. Eaedem solutiones reperiuntur ponendo generalius $z = XY$, vnde fit

$$\frac{X\,d^4 Y}{d y^4} = \frac{a a Y\,d d X}{d x^2},$$

qua aequatione ita repraesentata

$$\frac{d^4 Y}{Y\,d y^4} = \frac{a a\,d d X}{X\,d x^2},$$

vtrumque membrum eidem constanti aequari debet.

Scholion.

415. Aequatio autem ad quam totum negotium reduximus

$$\left(\frac{d d z}{d y^2}\right) = b\left(\frac{d z}{d x}\right)$$

ex earum est numero, quae nullo modo in genere resolui posse videntur, ita vt in solutionibus particularibus acquiescere debamus. Aequatio autem proposita non in mera speculatione est posita, sed quando laminarum elasti-

elasticarum vibrationes quam minimae in genere investigantur; ad huiusmodi aequationem quarti gradus resoluendam peruenitur, quae etiam causa est quod haec quaestio, non perinde atque cordarum vibrantium in genere adhuc resolui potuerit. Simili autem modo facile intelligitur hanc aequationem quarti gradus

$$\left(\frac{d^4 z}{dy^4}\right) = aa\left(\frac{ddz}{dx^2}\right) + 2ab\left(\frac{dz}{dx}\right) + bbz$$

reduci ad hanc geminatam secundi gradus

$$\left(\frac{ddz}{dy^2}\right) = \pm a\left(\frac{dz}{dx}\right) \pm bz$$

neque difficile est alios casus a posteriori eruere, vbi huiusmodi reductio ad gradum inferiorem locum inueniunt.

CAPVT III.

DE INTEGRATIONE AEQVATIONVM HOMOGENEARVM VBI SINGVLI TERMINI FORMVLAS DIFFERENTIALES EIVSDEM GRADVS CONTINENT.

Problema 69.

416.

Aequationis homogeneae ſecundi gradus

$$A\left(\frac{ddz}{dx^2}\right)+B\left(\frac{ddz}{dxdy}\right)+C\left(\frac{ddz}{dy^2}\right)=0$$

integralem, ſeu indolem functionis z inueſtigare, denotantibus litteris A, B, C quantitates quascunque conſtantes.

Solutio.

Hanc aequationem voco homogeneam, quia formulis differentialibus ſecundi gradus conſtat, neque praeterea alias quantitates variabiles inuoluit. Ad hanc reſoluendam obſeruo ei ſatisfacere huiusmodi aequationem homogeneam primi gradus:

$$\left(\frac{dz}{dx}\right)+\alpha\left(\frac{dz}{dy}\right)=\Delta=\text{Conſt.}$$

hac

hac enim duplici modo per x et y differentiata oritur:

$$\text{I. } \left(\frac{ddz}{dx^2}\right)+\alpha\left(\frac{ddz}{dx\,dy}\right)=0$$

$$\text{II. } \left(\frac{ddz}{dx\,dy}\right)+\alpha\left(\frac{ddz}{dy^2}\right)=0.$$

Iam illa per A hac vero per $\frac{C}{\alpha}$ multiplicata iunctim propositam producent si fuerit

$$A\alpha+\frac{C}{\alpha}=B \text{ seu}$$

$$A\alpha\alpha-B\alpha+C=0,$$

vnde duplex valor pro α resultat, quorum vterque per aequationem assumtam dabit partem functionis quaesitae z. Cum igitur sit

$$\left(\frac{dz}{dx}\right)=\Delta-\alpha\left(\frac{dz}{dy}\right) \text{ erit}$$

$$dz=\Delta dx+(dy-\alpha dx)\left(\frac{dz}{dy}\right),$$

patet $\left(\frac{dz}{dy}\right)$ functionem esse debere ipsius $y-\alpha x$, qua posita $=\Gamma':(y-\alpha x)$ erit

$$z=fx+\Gamma:(y-\alpha x)$$

denotante f constantem quamcunque. Quocirca aequationis propositae solutio ita se habebit. Formetur primo aequatio algebraica:

$$Auu+Bu+C=0$$

cuius factores simplices sint

$$u+\alpha \text{ et } u+\beta,$$

 ita

ita vt sit

$$Auu+Bu+C=A(u+\alpha)(u+\beta)$$

tum integrale quaesitum erit

$$z=fx+\Gamma:(y-\alpha x)+\Delta:(y-\beta x)$$

vbi cum prima pars fx iam in binis functionibus indefinitis contineri sit censenda ob

$$fx=\frac{f(y-\alpha x)-f(y-\beta x)}{\beta-\alpha}$$

succcinctius ita exprimetur

$$z=\Gamma:(y-\alpha x)+\Delta:(y-\beta x)$$

quod ob binas functiones arbitrarias vtique pro completo est habendum: vnico casu excepto, quo est $\beta=\alpha$. Pro quo casu statuamus $\beta=\alpha+d\alpha$, et cum sit

$$\Delta:(y-(\alpha+d\alpha)x)=\Delta:(y-\alpha x)-xd\alpha\,\Delta':(y-\alpha x)$$

quia pars prior iam in membro priori continetur, et loco posterioris scribere licet $x\Delta:(y-\alpha x)$ erit pro casu $\beta=\alpha$ seu $BB=4AC$ integrale

$$z=\Gamma:(y-\alpha x)+x\Delta:(y-\alpha x).$$

Coroll. 1.

417. Pro casu $\beta=\alpha$ manifestum est integrale etiam hoc modo exprimi posse:

$$z=\Gamma:(y-\alpha x)+y\Delta:(y-\alpha x)$$

quae autem forma ab illa non discrepat.

Coroll. 2.

418. Si $C=0$ vt sit

$$A\left(\frac{ddz}{dx^2}\right)+B\left(\frac{ddz}{dxdy}\right)=0,$$

hinc-

hincque

$$Auu+Bu=Au(u+\tfrac{B}{A}),\ \text{fit}$$

$$\alpha=0 \quad \text{et} \quad \beta=\tfrac{B}{A},$$

et integrale

$$z=\Gamma:y+\Delta:(y-\tfrac{B}{A}x)=\Gamma:y+\Delta:(Ay-Bx)$$

fimili modo aequationis

$$B(\tfrac{ddz}{dxdy})+C(\tfrac{ddz}{dy^2})=0$$

integrale eft

$$z=\Gamma:x+\Delta:(Cx-By).$$

Coroll. 3.

419. Porro huius aequationis:

$$aa(\tfrac{ddz}{dx^2})+2ab(\tfrac{ddz}{dxdy})+bb(\tfrac{ddz}{dy^2})=0 \ \text{ob}$$

$$aauu+2abu+bb=aa(u+\tfrac{b}{a})^2$$

eft integrale

$$z=\Gamma:(ay-bx)+x\Delta:(ay-bx).$$

Scholion.

420. Harum integralium forma nulla laborat difficultate quamdiu aequatio

$$Auu+Bu+C=0$$

duas habet radices reales fiue fint inaequales fiue aequales; quando autem hae radices fiunt imaginariae vt sit

$$\alpha = \mu + \nu\sqrt{-1} \text{ et } \beta = \mu - \nu\sqrt{1},$$

tum functiones arbitrariae omni fere vsu destituuntur. Etsi enim indoles functionum Γ et Δ lineis curuis vtcunque ductis repraesentatur, vt $\Gamma:v$ et $\Delta:v$ denotent in iis applicatas abscissae v conuenientes nullo modo patet, quomodo valores

$$\Gamma:(p+q\sqrt{-1}) \text{ et } \Delta:(p-q\sqrt{1})$$

exhiberi debeant, etiamsi imaginaria se mutuo tollant. In quo ingens cernitur discrimen inter functiones continuas et discontinuas, cum in illis semper valores ita expressi

$$\Gamma:(p+q\sqrt{-1})+\Gamma:(p-q\sqrt{-1}) \text{ et}$$

$$\frac{\Delta:(p+q\sqrt{-1})-\Delta:(p-q\sqrt{-1})}{\sqrt{-1}}$$

realiter exhiberi queant, id quod si Γ et Δ significent functiones discontinuas nullo modo succedit. His igitur casibus solutio generalis hic inuenta ad solas functiones continuas restringenda videtur, quandoquidem discontinuae applicationi et executioni aduersantur.

Proble-

Problema 70.

421. Propoſita hac aequatione tertii gradus homogenea:

$$A\left(\frac{d^3 z}{dx^3}\right)+B\left(\frac{d^3 z}{dx^2 dy}\right)+C\left(\frac{d^3 z}{dx\, dy^2}\right)+D\left(\frac{d^3 z}{dy^3}\right)=0$$

eius integrale completum inuenire.

Solutio.

Huic quoque aequationi, vti in praecedente problemate, ſatisfacere aequationem differentialem ſimplicem primi gradus, ſatis luculenter perſpicitur, ex quo integrale particulare talem habebit formam

$$z=\Gamma:(y+nx),$$

colligantur hinc ſingulae formulae differentiales tertii gradus, quae erunt

$$\left(\frac{d^3 z}{dx^3}\right)=+n^3\Gamma''':(y+nx);\ \left(\frac{d^3 z}{dx^2 dy}\right)=n^2\Gamma''':(y+nx)$$

$$\left(\frac{d^3 z}{dx\, dy^2}\right)=+n\Gamma''':(y+nx);\ \left(\frac{d^3 z}{dy^3}\right)=\Gamma''':(y+nx)$$

quibus ſubſtitutis, quoniam diuiſio per

$$\Gamma''':(y+nx)$$

ſuccedit naſcitur iſta aequatio:

$$An^3+Bn^2+Cn+D=0$$

cuius tres radices ſi fuerint $n=\alpha$, $n=\beta$, $n=\gamma$, euidens eſt, aequationi propoſitae ſatisfacere hanc formam

$$z=\Gamma:(y+\alpha x)+\Delta:(y+\beta x)+\Sigma:(y+\gamma x)$$

quae

quae cum tres functiones arbitrarias complectatur, dubium non est, quin ea sit integrale completum. Hoc tantum notetur, si duae radices sint aequales puta $\gamma = \beta$, integrale fore:

$$z = \Gamma:(y+\alpha x) + \Delta:(y+\beta x) + x\Sigma:(y+\beta x)$$

sin autem adeo omnes tres fuerint inter se aequales: $\gamma = \beta = \alpha$, tum erit integrale quaesitum:

$$z = \Gamma:(y+\alpha x) + x\Delta:(y+\alpha x) + xx\Sigma:(y+\alpha x).$$

Quodsi duae radices fuerint imaginariae, eadem erunt tenenda, quae modo ante sunt obseruata.

Coroll. 1.

422. Vltimus casus, quo tres radices sunt aequales, etiam inde est manifestus, quod si loco variabilium x et y binae nouae

$$t = x \quad \text{et} \quad u = y + \alpha x$$

introducantur, aequatio proposita contrahatur in hanc formam $\left(\frac{d^3 z}{dt^3}\right) = 0$, cuius integrale manifesto est

$$z = \Gamma:u + x\Delta:u + xx\Sigma:u.$$

Coroll. 2.

423. Hinc ergo etiam intelligitur, quomodo in aequationibus homogeneis altioris gradus si aequationes algebraicae inde formatae plures habeant radices aequales, integralia futura sint comparata.

Ita

Ita vt etiam tum neque casus radicum aequalium neque integralium vlli difficultati sit obnoxius.

Scholion.

424. Casus autem binarum radicum imaginariarum, quibus functiones arbitrariae nullum vsum habere videntur, ratione functionum continuarum, quae satisfaciunt, vberiorem euolutionem merentur. Formulae autem his casibus in integrale ingredientes semper ad hanc formam reduci possunt:

$$\Gamma : v(\cos.\Phi + \sqrt{-1}.\sin.\Phi) + \Delta : v(\cos.\Phi - \sqrt{-1}.\sin.\Phi)$$

vnde primum si functiones sint potestates, huiusmodi valores colliguntur:

$$A v^n \cos. n\Phi + B v^n \sin. n\Phi \text{ seu } A v^n \cos.(n\Phi + \alpha)$$

quotcunque enim huiusmodi valores, constantes A, n et α vtcunque mutando adhiberi possunt. Deinde si functiones denotent logarithmos, prodeunt tales valores:

$$A\, l v + B\Phi.$$

Tertio si functiones sint exponentiales, oriuntur hi:

$$e^{v\cos.\Phi}(A\cos.(v\sin.\Phi) + B\sin.(v\sin.\Phi)) = A e^{v\cos.\Phi}\cos.(v\sin.\Phi + \alpha)$$

et generalius

$$A e^{v^n \cos. n\Phi} \cos.(v^n \sin. n\Phi + \alpha).$$

Plurimae autem aliae huiusmodi formulae ex doctrina imaginariorum elici possunt, quae vtcunque cum his combinatae, pro parte integrali ex binis radicibus imaginariis nata vſurpari poterunt, vnde infinita functionum multitudo naſcitur, quae ſolutionem completam mentiri videtur, neque tamen pro completa perinde haberi poteſt, atque vſu venit iis caſibus, quibus omnes radices ſunt reales. Hic autem obſeruetur, nullum adhuc problema mechanicum ſeu phyſicum occurriſſe, quod ab huiusmodi caſu penderet.

Problema 71.

425. Propoſita huiusmodi aequatione homogenea gradus cuiuscunque

$$A\left(\frac{d^\lambda z}{dx^\lambda}\right)+B\left(\frac{d^\lambda z}{dx^{\lambda-1}dy}\right)+C\left(\frac{d^\lambda z}{dx^{\lambda-2}dy^2}\right)+\text{etc.}=0$$

eius integrale completum inuenire.

Solutio.

Formetur hinc aequatio algebraica ordinis λ

$$An^\lambda+Bn^{\lambda-1}+Cn^{\lambda-2}+\text{etc.}=0$$

cuius radices numero λ ſint:

$$n=\alpha,\ n=\beta,\ n=\gamma,\ n=\delta \text{ etc.}$$

quae ſi omnes fuerint inaequales, integrale completum aequationis propoſitae erit

$$z=\Gamma:(y+\alpha x)+\Delta:(y+\beta x)+\Sigma:(y+\gamma x)$$
$$+\Theta:(y+\delta x) \text{ etc.}$$

quarum functionum diſparium numerus erit $=\lambda$. Sin autem eueniat, vt inter has radices duae pluresue

resue reperiantur aequales, scilicet $\beta=\alpha$, $\gamma=\alpha$, tum functiones has radices aequales inuoluentes respectiue multiplicari debent per terminos progressionis geometricae huius $1, x, x^2$ etc. vel huius $1, y, y^2$ etc. ita vt functionum arbitrariarum numerus non minuatur. De radicibus autem imaginariis perpetuo ea sunt notanda quae ante obseruauimus, nisi forte functiones arbitrarias formularum imaginariarum excludere nolimus.

Coroll. 1.

426. Casu radicum aequalium perinde est, vtra serie geometrica vtamur, siquidem functiones neque sint ipsius x neque ipsius y tantum. Sin autem hae functiones fuerint vel ipsius x vel ipsius y tantum tum alterius variabilis diuersae progressione geometrica vti oportet.

Coroll. 2.

427. Si in aequatione algebraica termini initiales A, B, C etc. euanescant, vt radicum numerus exponente λ minor esse videatur, tum radices deficientes pro infinite magnis sunt habendae, quibus functiones ipsius x tantum respondebunt, in integrale introducendae.

Coroll. 3.

428. Ita si fuerit $A=0$, $B=0$ et $C=0$, tres radices α, β, γ in infinitum excrescere sunt censendae, ex quibus nascetur pars integralis:

$$\Gamma:x+y\,\Delta:x+y^2+\Sigma:x.$$

Scholion.

429. Quoniam haec pars calculi integralis vix excoli coepit, ideoque huius generis inueſtigationes adhuc prorſus ſunt reconditae, de hac ſectione plura proferre non licet, ideoque his partem primam libri ſecundi, quae in inueſtigatione functionum binarum variabilium ex data quadam differentialium relatione verſatur, concludere cogor. Multo autem pauciora circa partem alteram huius libri in medium afferre conceditur, vbi calculus integralis ad functiones trium variabilium accommodatur, hancque ob cauſam ne operae quidem erit pretium iſtam partem in ſectiones ſubdiuidere multo minus ſequentes partes attingere.

CALCVLI INTEGRALIS
LIBER POSTERIOR.

PARS ALTERA

INVESTIGATIO FVNCTIONVM TRIVM VARIABILIVM EX DATA DIFFERENTIALIVM RELATIONE.

CAPUT I.

DE FORMULIS DIFFERENTIALIBUS FUNCTIONUM TRES VARIABILES INVOLVENTIUM.

Problema 72.

430.

Si v sit functio quaecunque trium quantitatum variabilium x, y et z, eius formulas differentiales primi gradus exhibere.

Solutio.

Cum v sit functio trium variabilium x, y et z, si ea more solito differentietur, eius differentiale in genere ita reperietur expressum:

$$dv = pdx + qdy + rdz.$$

Tribus

Tribus scilicet id constabit partibus, quarum prima pdx seorsim inuenitur, si in differentiatione sola quantitas x vt variabilis tractetur, binis reliquis y et z vt constantibus spectatis. Simili modo pars secunda qdy impetratur differentiatione functionis v ita instituta vt sola quantitas y pro variabili, binae reliquae vero x et z pro constantibus habeantur, quod idem de parte tertia rdz est tenendum, quae est differentiale ipsius v variabilitatis solius quantitatis z ratione habita. Hinc patet, quomodo per differentiationem quantitates istae p, q et r seorsim sint inueniendae, quas hic formulas differentiales primi gradus functionis v appellabo, et ne nouis litteris in calculum introducendis sit opus, eas naturae suae conuenienter ita indicabo:

$$p = \left(\frac{dv}{dx}\right);\ q = \left(\frac{dv}{dy}\right);\ r = \left(\frac{dv}{dz}\right).$$

Quaelibet ergo functio v trium variabilium x, y et z tres habet formulas differentiales primi gradus ita designandas

$$\left(\frac{dv}{dx}\right);\ \left(\frac{dv}{dy}\right);\ \left(\frac{dv}{dz}\right),$$

in quarum qualibet vnicae variabilis ratio habetur, dum binae reliquae vt constantes spectantur, et quoniam differentialia per diuisionem tolluntur, hae formulae differentiales ad classem quantitatum finitarum sunt referendae.

Coroll. 1.

Coroll. 1.

431. Ex tribus formulis differentialibus functionis v inuentis eius differentiale solito more sumtum ita conflatur, vt sit

$$dv = dx\left(\frac{dv}{dx}\right) + dy\left(\frac{dv}{dy}\right) + dz\left(\frac{dv}{dz}\right);$$

cuius ergo formae viciſſim integrale eſt ipſa illa functio v, vel etiam eadem quantitate quacunque ſiue aucta ſiue minuta.

Coroll. 2.

432. Si trium variabilium x, y et z functio v fuerit data eius formulae differentiales ſingulae

$$\left(\frac{dv}{dx}\right); \left(\frac{dv}{dy}\right); \left(\frac{dv}{dz}\right)$$

iterum erunt functiones certae earundem variabilium x, y et z per differentiationem facile inueniendae. Interim tamen euenire poteſt; vt vna pluresue variabilium ex huiusmodi formulis differentialibus prorſus excedant.

Scholion 1.

433. Nihil etiam impedit, quominus quantitas v vt functio trium variabilium x, y et z ſpectari poſſit, etiamſi forte duas tantum inuoluat, dum ſcilicet ratio compoſitionis ita eſt comparata, vt tertia quaſi caſu exceſſerit; quod eo minus eſt

 miran-

mirandum, cum idem in functionibus tam vnius quam duarum variabilium euenire possit. Quoniam enim functiones vnius variabilis commodissime per applicatas cuiuspiam lineae curuae repraesentari solent, siquidem pro curuae natura applicatae eius vt certae functiones abscissae x spectari possunt casu quo linea curua abit in lineam rectam axi parallelam, etsi tum applicata quantitati constanti aequatur, propterea tamen ex illa idea generali, qua vt functio abscissae x spectatur, neutiquam excluditur, neque enim si quaeratur, qualis sit functio y ipsius x? incongrue is respondere est censendus, qui dicat hanc functionem y aequari quantitati constanti. Quod deinde ad functiones binarum variabilium x et y attinet, quas semper per interualla, quibus singula cuiusdam superficiei puncta a quopiam plano distant, repraesentare licet, dum binae variabiles x et y in hoc plano accipiuntur, manifestum est vtique superficiem ita comparatam esse posse, vt functio illa, vel per solam x vel per solam y determinetur. Quin etiam si superficies fuerit plana ipsique illi plano parallela, functio illa adeo abit in quantitatem constantem; neque propterea minus tanquam functio binarum variabilium considerari debebit. Quamobrem etiam quando tractatio circa functiones trium variabilium versatur, in eo genere etiam eiusmodi functiones, quae tantum vel per binas vel vnicam trium variabilium x, y et z determinantur, vel adeo ipsae sunt quantitates constantes.

Scho-

Scholion 2.

434. In calculo differentiali iam eſt oſtenſum, functionum plures variabiles inuoluentium differentialia inueniri, ſi vnaquaeque variabilium ſeorſim tanquam ſola eſſet variabilis ſpectetur, atque omnia differentialia inde nata in vnam ſummam coniiciantur. Quodſi ergo differentiatio hoc modo inſtituatur, ſingulae iſtae operationes, deleto tantum differentiali, praebebunt formulas differentiales, quas his ſignis

$$\left(\frac{dv}{dx}\right),\ \left(\frac{dv}{dy}\right) \text{ et } \left(\frac{dv}{dz}\right)$$

indicamus: ſimulque intelligitur, quomodo etiam functionum quatuor pluresue variabiles inuoluentium formulae differentiales ſint inueniendae. Circa functiones autem trium variabilium x, y et z exempla aliquot ſubiungamus, quibus earum ternas formulas differentiales exhibebimus.

Exemplum 1.

435. *Si functio trium variabilium ſit*

$$v = \alpha x + \beta y + \gamma z$$

eius formulae differentiales ita ſe habebunt.

Cum per differentiationem prodeat

$$dv = \alpha dx + \beta dy + \gamma dz$$

manifeſtum eſt fore:

$$\left(\frac{dv}{dx}\right) = \alpha;\ \left(\frac{dv}{dy}\right) = \beta;\ \left(\frac{dv}{dz}\right) = \gamma$$

Ddd 2 ſicque

ſicque omnes tres formulas differentiales eſſe conſtantes.

Exemplum 2.

436. *Si functio trium variabilium ſit*

$$v = x^{\lambda} y^{\mu} z^{\nu},$$

eius formulae differentiales ita ſe habebunt.

Differentiatione more ſolito peracta fit:

$$dv = \lambda x^{\lambda-1} y^{\mu} z^{\nu} dx + \mu x^{\lambda} y^{\mu-1} z^{\nu} dy + \nu x^{\lambda} y^{\mu} z^{\nu-1} dz$$

vnde perſpicuum eſt fore formulas differentiales:

$$\left(\frac{dv}{dx}\right) = \lambda x^{\lambda-1} y^{\mu} z^{\nu}; \quad \left(\frac{dv}{dy}\right) = \mu x^{\lambda} y^{\mu-1} z^{\nu}; \quad \left(\frac{dv}{dz}\right) = \nu x^{\lambda} y^{\mu} z^{\nu-1}$$

quae ergo ſingulae ſunt nouae functiones omnium trium variabilium x, y, z niſi exponentes λ, μ, ν ſint vel nihilo vel vnitati aequales.

Exemplum 3.

437. *Si functio* v *duas tantum inuoluat variabiles* x *et* y, *tertia* z *in eius compoſitionem non ingrediente, formulae differentiales ita ſe habebunt.*

Quia functio v duas tantum variabiles x et y implicat, eius differentiale huiusmodi formam induet:

$$dv = p\,dx + q\,dy + o\,dz$$

tertia ſcilicet parte ex variabilitate ipſius z orta euaneſcente, vnde habebimus:

$$\left(\frac{dv}{dx}\right) = p; \quad \left(\frac{dv}{dy}\right) = q \quad \text{et} \quad \left(\frac{dv}{dz}\right) = o.$$

Corol-

Corollarium.

438. Hinc ergo viciſſim patet, ſi fuerit $(\frac{dv}{dz})=0$ tum fore v functionem quamcunque binarum variabilium x et y, quam in poſterum ita indicabimus $v=\Gamma:(x,y)$ denotante $\Gamma:(x\ y)$ functionem quamcunque binarum variabilium x et y.

Scholion.

439. Mox oſtendemus, quando functio trium variabilium ex data quadam relatione ſeu conditione formularum differentialium inueſtiganda proponitur, qualibet integratione introduci functionem quamcunque arbitrariam binarum variabilium, atque adeo in hoc conſiſtere criterium, quo haec pars calculi integralis a praecedentibus diſtinguitur. Quemadmodum enim, dum natura functionum vnicae variabilis ex data differentialium conditione inueſtigatur, in quo vniuerſus liber primus eſt occupatus, per quamlibet integrationem quantitas conſtans arbitraria in calculum inuehitur, ita in parte praecedente huius ſecundi libri vidimus, ſi functiones binarum variabilium ex data formularum differentialium relatione inueſtigari debeant, tum ad eſſentiam huius tractationis id pertinere, quod qualibet integratione non quantitas conſtans ſed adeo functio vnius variabilis prorſus arbitraria in calculum introducatur; etſi enim plerumque hae functiones veluti $\Gamma:(\alpha x+\beta y)$ ambas variabiles x et y implicabant, tamen ibi tota

quantitas $\alpha x + \beta y$ vt vnica ſpectatur, cuius functionem quamcunque illa formula $\Gamma : (\alpha x + \beta y)$ denotat. Nunc igitur, vbi de functionibus trium variabilium agitur probe notandum eſt, qualibet integratione functionem arbitrariam duarum adeo variabilium in calculum introduci: ex quo ſimul indolem integrationum, quae circa functiones plurium variabilium verſantur, colligere licet.

Problema 73.

440. Si ſit v functio quaecunque trium variabilium x, y et z eius formulas differentiales ſecundi altiorumque graduum exhibere.

Solutio.

Cum eius formulae differentiales primi gradus ſint tres

$$\left(\frac{dv}{dx}\right),\ \left(\frac{dv}{dy}\right),\ \left(\frac{dv}{dz}\right),$$

quaelibet inſtar nouae functionis conſiderata iterum tres ſuppeditabit formulas differentiales, quae autem ob

$$\left(\frac{ddv}{dx\,dy}\right) = \left(\frac{ddv}{dy\,dx}\right)$$

reducentur ad ſex ſequentes:

$$\left(\frac{ddv}{dx^2}\right);\ \left(\frac{ddv}{dy^2}\right);\ \left(\frac{ddv}{dz^2}\right);\ \left(\frac{ddv}{dx\,dy}\right);\ \left(\frac{ddv}{dy\,dz}\right);\ \left(\frac{ddv}{dx\,dz}\right)$$

ex quarum denominatoribus intelligitur, quaenam trium quantitatum x, y, z in vtraque differentiatione pro ſola variabili haberi debeat. Simili modo

cui-euidens eſt formulas differentiales tertii gradus dari decem ſequentes:

$$\left(\frac{d^3 v}{dx^3}\right);\ \left(\frac{d^3 v}{dx^2 dy}\right);\ \left(\frac{d^3 v}{dx\, dy^2}\right)$$
$$\left(\frac{d^3 v}{dy^3}\right);\ \left(\frac{d^3 v}{dy^2 dz}\right);\ \left(\frac{d^3 v}{dy\, dz^2}\right);\ \left(\frac{d^3 v}{dx\, dy\, dz}\right)$$
$$\left(\frac{d^3 v}{dz^3}\right);\ \left(\frac{d^3 v}{dz^2 dx}\right);\ \left(\frac{d^3 v}{dz\, dx^2}\right).$$

Formularum porro differentialium quarti gradus numerus eſt, 15 quinti 21 etc. ſecundum numeros triangulares, ſimulque ex cuiusque forma perſpicuum eſt, quomodo eius valor ex data functione v per repetitam differentiationem, in qualibet vnicam variabilem conſiderando elici debeat.

Coroll. 1.

441. En ergo omnes formulas differentiales cuiusque gradus, quas ex qualibet functione trium variabilium deriuare licet per differentiationem, quae porro vt functiones trium variabilium ſpectari poſſunt.

Coroll. 2.

442. Quemadmodum ergo ex huiusmodi functione data omnes eius formulae differentiales ope calculi differentialis inueniuntur, ita viciſſim ex data quapiam formula differentiali, vel duarum pluriumue relatione quadam ope calculi integralis ipſa illa functio vnde eae naſcuntur, inueſtigari debet.

Scho-

Scholion 1.

443. In calculo quidem differentiali parum refert, vtrum functio differentianda vnam pluresue variabiles inuoluat, cum praecepta differentiandi pro quouis variabilium numero maneant eadem; quam ob caussam etiam calculum differentialem secundum hanc functionum varietatem in diuersas partes distingui non erat opus. Longe secus autem accidit in calculo integrali, quem secundum hanc functionum varietatem omnino in partes diuidi necesse est, quippe quae partes tam ratione propriae indolis quam ratione praeceptorum maxime inter se discrepant. Quemadmodum igitur hanc partem circa functiones trium variabilium occupatam tractari conueniat, exponendum videtur. Ac primo quidem ii casus commodissime euoluentur, quibus vnius cuiusdam formulae differentialis valor datur, ex quo indolem functionis quaesitae definiri oporteat, quoniam haec inuestigatio nulla laborat difficultate. Deinde huiusmodi quaestiones aggrediar, quibus relatio quaepiam inter duas pluresue formulas differentiales proponitur; vbi quidem plurimum refert, cuiusnam gradus ea fuerint, siquidem ex primo gradu plures casus expedire licet, dum ex altioribus vix adhuc quicquam in medium afferri potest: hunc ergo ordinem in ista tractatione obseruabo.

Scho-

Scholion 2.

444. Videri hic poſſet ad functiones trium variabilium definiendas duas adeo conditiones ſeu relationes inter formulas differentiales admitti poſſe, neque vnica praeſcripta quaeſtionem eſſe determinatam. Quodſi enim ponatur

$$dv = pdx + qdy + rdz,$$

vbi litterae p, q, r vicem gerunt formularum differentialium primi gradus, atque verbi gratia hae duae proponantur conditiones vt ſit

$$q = p \quad \text{et} \quad r = p$$

ac propterea

$$dv = p(dx + dy + dz),$$

manifeſtum eſt ſolutionem dari poſſe ſcilicet

$$v = \Gamma : (x + y + z).$$

Verum ad hanc obiectionem reſpondeo, in hoc exemplo caſu euenire, vt binae conditiones ſimul conſiſtere poſſint, altera enim parumper immutata vt manente $q = p$ eſſe debeat $r = px$ ideoque

$$dv = p(dx + dy + xdz),$$

perſpicuum eſt, nullum pro p valorem exhiberi poſſe, per quem formula differentialis

$$dx + dy + xdz$$

multiplicata integrabilis reddatur, quod vnicum exemplum ſufficit ad demonſtrandum, duabus conditionibus praeſcribendis huiusmodi quaeſtiones euadere plusquam determinatas, neque propterea ſolutionem admittere niſi certis caſibus quibus quaſi altera conditio iam in altera inuoluitur. Quocirca ſemper vnica relatio inter formulas differentiales propoſito omnino ſufficit problemati determinando, quod idcirco, quia per integrationem functio arbitraria indefinita ingreditur, aeque parum pro indeterminato eſt habendum ac problemata calculi integralis communis quorum ſolutio conſtantem arbitrariam introducit.

CAPVT

CAPVT II.

DE INVENTIONE FVNCTIONVM TRIVM VARIABILIVM EX DATO CVIVSPIAM FORMVLAE DIFFERENTIALIS VALORE.

Problema 74.

445.

Dato valore cuiuspiam formulae differentialis primi gradus, inuestigare ipsam functionem trium variabilium, ex qua illa formula differentialis nascitur.

Solutio.

Sit v functio quaesita trium variabilium x, y et z et S earundem functio data quaecunque, cui formula differentialis $\left(\frac{dv}{dx}\right)$ debeat esse aequalis. Cum igitur sit $\left(\frac{dv}{dx}\right)=S$, erit posita sola quantitate x varibili binis reliquis vero y et z vt constantibus spectatis $dv=Sdx$ ideoque

$$v=\int Sdx+\text{Const.}$$

vbi notandum est in integratione formulae Sdx ambas quantitates y et z pro constantibus haberi, et

 loco

loco *Const.* functionem quamcunque ipsarum y et z scribi debere, ex quo functio quaesita ita exhiberi poterit:

$$v = \int S\,dx + T:(y \text{ et } z)$$

hic scilicet $T:(y \text{ et } z)$ quantitatem quamcunque ex binis quantitatibus y, et z vna cum constantibus vtcunque conflatam denotat.

Simili modo si proponatur $\left(\frac{dv}{dy}\right) = S$ erit

$$v = \int S\,dy + T:(x \text{ et } z)$$

et haec aequatio $\left(\frac{dv}{dz}\right) = S$ integrata praebet

$$v = \int S\,dz + T:(x \text{ et } y).$$

Coroll. 1.

446. Hic iam abunde intelligitur integratione huiusmodi functionum loco constantis introduci functionem arbitrariam duarum quantitatum variabilium, atque adeo in hoc characterem harum integrationum esse constituendum.

Coroll. 2.

447. Hic ergo istud problema solutum dedimus, quo quaeritur functio v trium variabilium x, y, z, vt posito

$$dv = p\,dx + q\,dy + r\,dz,$$

fiat vel $p = S$, vel $q = S$, vel $r = S$, existente S functio-

functione quacunque data easdem variabiles, vel duas, vel vnicam inuoluente.

Coroll. 3.

448. Quodsi igitur esse debeat $(\frac{dv}{dx}) = 0$, seu $p = 0$, functio quaesita erit $v = \Gamma:(y \text{ et } z)$, et vt fiat $(\frac{dv}{dy}) = 0$ erit $v = \Gamma:(x \text{ et } z)$, tum vero vt fiat $(\frac{dv}{dz}) = 0$, necesse est sit $v = \Gamma:(x \text{ et } y)$.

Scholion 1.

449. Quemadmodum in praecedente parte functiones arbitrariae vnius variabilis per applicatas curuarum quarumcunque siue regularium siue etiam irregularium repraesentari poterant, ita in hac parte functiones binarum variabilium arbitrariae per superficiem pro lubitu descriptam repraesentari possunt. Ita si super plano, in quo binae coordinatae x et y more solito assumuntur, superficies quaecunque expansa concipiatur, tertia coordinata distantiam cuiusvis superficiei puncti ab illo plano designans, functionem quamcunque binarum variabilium x et y repraesentabit. Hocque modo aptissime vera idea huiusmodi functionum constitui videtur, cum ex ea non solum ratio harum functionum regularium sed etiam irregularium perspiciatur.

Scholion 2.

450. Hic etiam notari conuenit huiusmodi functiones binarum variabilium infinitis diuersis mo-

dis etiam designari posse. Variatis enim in plano memorato binis coordinatis x et y, in binas alias t et u, vt sit $t = \alpha x + \beta y$ et $u = \gamma x + \delta y$, manifestum est functionem binarum variabilium t et u seu $\Gamma:(t \text{ et } u)$ conuenire cum functione ipsarum x et y seu $\Gamma:(x \text{ et } y)$; si enim loco t et u illi valores pro x et y substituantur vtique prodit functio duas tantum variabiles x et y inuoluens. Atque multo generalius si t aequetur functioni cuipiam datae ipsarum x et y, pariterque u huiusmodi alii functioni, tum $\Gamma:(t \text{ et } u)$ facta substitutione abibit in functionem ipsarum x et y ita exprimendam $\Delta:(x \text{ et } y)$; non enim necesse est vt idem functionis character Γ rationem compositionis quasi denotans vtrinque sit idem cum hic in genere de functionibus quibuscunque agatur. Quare si in sequentibus forte eiusmodi functiones occurrant:

$\Gamma:(ax+by \text{ et } fxx+gyy)$, vel $\Gamma:(\sqrt{(xx+yy)} \text{ et } l\frac{x}{y})$ etc.

earum loco semper haec forma simplex $\Gamma:(x \text{ et } y)$ scribi potest.

Scholion 3.

451. Solutionis, quam dedimus, consideratio nobis suppeditat sequentes reflexiones. Primo posito

$$dv = p\,dx + q\,dy + r\,dz$$

si debeat esse $p = (\frac{dv}{dx}) = 0$, fiet

$$dv = q\,dy + r\,dz,$$

vnde patet v eiusmodi esse quantitatem, cuius differenferentiale hanc habiturum ſit formam $q dy + r dz$; quod fieri nequit, niſi quantitas v fuerit functio binarum variabilium y et z tantum, tertia x penitus excluſa; et quia circa quantitates q et r nulla conditio praeſcribitur, recte pronunciamus, loco quantitatis v accipi poſſe functionem quamcunque binarum variabilium y et z ſeu eſſe $v = \Gamma:(y \text{ et } z)$, quam eandem ſolutionem conſideratio formulae $(\frac{dv}{dx}) = 0$ ſuggeſſit. Deinde ſi eſſe debeat generalius $(\frac{dv}{dx}) = p = S$ denotante S quantitatem quamcunque ex variabilibus x, y, z conflatam, habebimus

$$dv = S dx + q dy + r dz$$

quae aequatio ita reſoluitur. Quaeratur primo integrale formulae $S dx$ ſola quantitate x vt variabili ſpectata, quod ſit $= V$; haecque quantitas per omnes tres variabiles differentiata praebeat

$$dV = S dx + Q dy + R dz,$$

ex quo cum ſit

$$S dx = dV - Q dy - R dz \text{ erit}$$

$$dv = dV + (q - Q) dy + (r - R) dz \text{ ſeu}$$

$$d.(v - V) = (q - Q) dy + (r - R) dz,$$

vnde vt ante patet quantitatem $v - V$ functioni cuicunque binarum variabilium y et z, aequari poſſe. Quare ob $V = \int S dx$, prodit vt ante

$$v = \int S dx + \Gamma:(y \text{ et } z);$$

hocque ratiocinium, quo iſthuc peruenimus, diligenter

genter notari meretur, cum etiam in parte prima eximium vſum praeſtare poſſit. Propoſita enim aequatione

$$\left(\frac{ddz}{dy^2}\right)=aa\left(\frac{ddz}{dx^2}\right),$$

quia eſt

$$d.\left(\frac{dz}{dx}\right)=dx\left(\frac{ddz}{dx^2}\right)+dy\left(\frac{ddz}{dx\,dy}\right) \text{ et}$$

$$d.\left(\frac{dz}{dy}\right)=dx\left(\frac{ddz}{dx\,dy}\right)+dy\left(\frac{ddz}{dy^2}\right)$$

erit:

$$a\,d.\left(\frac{dz}{dx}\right)+d.\left(\frac{dz}{dy}\right)=\left(\frac{ddz}{dx^2}\right)(a\,dx+aa\,dy)$$
$$+\left(\frac{ddz}{dx\,dy}\right)(a\,dy+dx)$$

ſeu

$$ad.\left(\frac{dz}{dx}\right)+d.\left(\frac{dz}{dy}\right)=(dx+a\,dy)\left(a\left(\frac{ddz}{dx^2}\right)+\left(\frac{ddz}{dx\,dy}\right)\right)$$

cuius poſterioris membri integrale manifeſto eſt $F:(x+ay)$ hincque

$$\left(\frac{dz}{dy}\right)=-a\left(\frac{dz}{dx}\right)+a\Gamma':(x+ay),$$

quo vna integratio abſoluta eſt cenſenda. Quare cum ſit

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right)$$

habebitur

$$dz=\left(\frac{dz}{dx}\right)(dx-a\,dy)+a\,dy\,\Gamma':(x+ay).$$

Sit $\left(\frac{dz}{dx}\right)=p$ et $x-ay=t$,

vt fiat

$$dz=p\,dt+a\,dy\,\Gamma':(t+2ay),$$

pro

pro duabus variabilibus t et y hincque

$$z = \tfrac{1}{2}\Gamma:(t+2ay) + \int dt\,(p - \tfrac{1}{2}\Gamma':(t+2ay))$$
$$= \Gamma:(x+ay) + \Delta:(x-ay)$$

quia

$$\Delta:t = \Delta:(x-ay) \text{ et } \Gamma:(t+2ay) = \Gamma:(x+ay).$$

Problema 75.

452. Inuestigare indolem functionis trium variabilium x, y, z cuius formula quaedam differentialis secundi gradus aequetur datae cuipiam functioni S.

Solutio.

Denotet v functionem quaesitam, et cum eius sex dentur formulae differentiales secundi gradus, ponamus primo esse debere $\left(\frac{ddv}{dx^2}\right) = S$, et integratione semel instituta prodit

$$\left(\frac{dv}{dx}\right) = \int S\,dx + \Gamma:(y \text{ et } z),$$

iterumque integrando

$$v = \int dx \int S\,dx + x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z)$$

vbi in formulae $\int dx \int S\,dx$ duplici integratione sola quantitas x vt variabilis spectatur, quemadmodum iam supra est inculcatum. Similis autem omnino est integratio aequationum

$$\left(\frac{ddv}{dy^2}\right) = S \text{ et } \left(\frac{ddv}{dz^2}\right) = S.$$

Pro reliquis formulis differentialibus secundi gradus sufficit hanc vnam $\left(\frac{ddv}{dxdy}\right)=S$ resoluisse; quae primo per solam variabilem x integrata dabit

$$\left(\frac{dv}{dy}\right)=\int S\,dx+f:(y \text{ et } z).$$

Deinde altera integratione per solam variabilem y instituta colligitur:

$$v=\int dy\int S\,dx+\int dy\,f:(y \text{ et } z)+\Delta:(x \text{ et } z)$$

vbi primum obseruo partem primam nullo discrimine ordinis inter binas variabiles x et y habito ita $\iint S\,dx\,dy$ exprimi posse. Deinde quaecunque fuerit $f:(y \text{ et } z)$ functio ipsarum y et z, si ea per dy multiplicetur et spectata z vt constante integretur, euidens est denuo functionem ipsarum y et z prodire, et quia illa nullo modo determinatur, etiam hanc fore indeterminatam ideoque arbitrariam, vnde statuere poterimus:

$$v=\iint S\,dx\,dy+\Gamma:(y \text{ et } z)+\Delta:(x \text{ et } z).$$

Coroll. 1.

453. Hic obseruo per integrationem formulae $\int dy\,f:(y \text{ et } z)$ iam sponte formulam $\Delta:(x \text{ et } z)$ inuehi; cum enim ibi sola quantitas y vt variabilis spectetur, loco quantitatis constantis per integrationem adiiciendae functio quaecunque ipsarum x et z scribi poterit.

Coroll. 2.

454. Quodsi functio illa data S euanescat, sequentes integrationes prouenient:

si $\left(\frac{ddv}{dx^2}\right)=0$, erit $v=x\Gamma:(y \text{ et } z)+\Delta:(y \text{ et } z)$

si $\left(\frac{ddv}{dy^2}\right)=0$, erit $v=y\Gamma:(x \text{ et } z)+\Delta:(x \text{ et } z)$

si $\left(\frac{ddv}{dz^2}\right)=0$, erit $v=z\Gamma:(x \text{ et } y)+\Delta:(x \text{ et } y)$

si $\left(\frac{ddv}{dx\,dy}\right)=0$, erit $v=\Gamma:(x \text{ et } z)+\Delta:(y \text{ et } z)$

si $\left(\frac{ddv}{dx\,dz}\right)=0$, erit $v=\Gamma:(x \text{ et } y)+\Delta:(y \text{ et } z)$

si $\left(\frac{ddv}{dy\,dz}\right)=0$, erit $v=\Gamma:(x \text{ et } y)+\Delta:(x \text{ et } z)$.

Coroll. 3.

455. Quia hic duplici opus est integratione, atque etiam duae functiones arbitrariae, vtraque binarum variabilium in calculum sunt inuectae; hoc certissimum est criterium haec integralia inuenta esse completa.

Scholion.

456. Alio etiam modo haec eadem integralia erui possunt, qui nititur principio supra (451.) indicato, quod si fuerit

$$dv=Sdx+qdy+rdz \text{ fore}$$

$$v=\int Sdx+f:(y \text{ et } z).$$

Secundum hoc principium ergo si fuerit $\left(\frac{ddv}{dx^2}\right)=S$ erit

$$d.\left(\frac{dv}{dx}\right)=S\,dx+dy\left(\frac{ddv}{dx\,dy}\right)+dz\left(\frac{ddv}{dx\,dz}\right),$$

qua forma cum illa collata loco v habemus $\left(\frac{dv}{dx}\right)$ et loco q et r has formulas

$$\left(\frac{ddv}{dx\,dy}\right) \text{ et } \left(\frac{ddv}{dx\,dz}\right),$$

ex quo integrale erit

$$\left(\frac{dv}{dx}\right)=\int S\,dx+f:(y \text{ et } z).$$

Cum iam porro sit

$$dv=\left(\frac{dv}{dx}\right)dx+\left(\frac{dv}{dy}\right)dy+\left(\frac{dv}{dz}\right)dz \text{ erit}$$

$$dv=dx\int S\,dx+dx f:(y \text{ et } z)+dy\left(\frac{dv}{dy}\right)+dz\left(\frac{dv}{dz}\right)$$

vnde pariter manifesto sequitur:

$$v=\int dx\int S\,dx+x\Gamma:(y \text{ et } z)+\Delta:(y \text{ et } z).$$

Pari modo operatio est instituenda pro aequatione $\left(\frac{ddv}{dx\,dy}\right)=S$, inde enim fit

$$d.\left(\frac{dv}{dy}\right)=S\,dx+dy\left(\frac{ddv}{dy^2}\right)+dz\left(\frac{ddv}{dy\,dz}\right)$$

cuius integrale est

$$\left(\frac{dv}{dy}\right)=\int S\,dx+f:(y \text{ et } z);$$

altera integratio instituatur in hac forma

$$dv=dy\int S\,dx+dy f:(y \text{ et } z)+dx\left(\frac{dv}{dx}\right)+dz\left(\frac{dv}{dz}\right)$$

vnde ob

$$\int dy f:(y \text{ et } z)=\Gamma:(y \text{ et } z)$$

obti-

obtinetur vt ante:

$$v = \int\int S\,dx\,dy + \Gamma:(y \text{ et } z) + \Delta:(x \text{ et } z).$$

Problema 76.

457. Inuestigare indolem functionis trium variabilium x, y et z, cuius quaedam formula differentialis tertii gradus aequetur datae cuipiam quantitati S ex illis variabilibus et constantibus vtcunque compositae.

Solutio.

Posita functione quaesita $= v$, percurramus non tam singulas eius formulas differentiales tertii gradus, quam eas quarum ratio est diuersa.

Sit igitur primo $(\frac{d^3 v}{dx^3}) = S$, et prima integratio statim dat

$$(\frac{ddv}{dx^2}) = \int S\,dx + 2\Gamma:(y \text{ et } z),$$

tum vero altera

$$(\frac{dv}{dx}) = \int dx \int S\,dx + 2x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z)$$

vnde tandem colligitur:

$$v = \int dx \int dx \int S\,dx + xx\Gamma:(y \text{ et } z) + x\Delta:(y \text{ et } z) + \Sigma:(y \text{ et } z).$$

Sit secundo $(\frac{d^3 v}{dx^2\,dy}) = S$ et binae priores integrationes vt ante dant:

$$(\frac{dv}{dy}) = \int dx \int S\,dx + x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z)$$

 quia

quia nunc vt vidimus pro $\int dy \Gamma:(y \text{ et } z)$ scribere licet: $\Gamma:(y \text{ et } z)$ per tertiam integrationem inuenimus:

$$v = \int^{2} S dx^{2} dy + x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z) + \Sigma:(x \text{ et } z).$$

In his autem duobus casibus omnes formulae differentiales tertii gradus, variabilibus permutandis, continentur, sola excepta vltima hac $\left(\frac{d^3 v}{dx\,dy\,dz}\right)$, quam idcirco seorsim tractari oportet:

Sit igitur $\left(\frac{d^3 v}{dx\,dy\,dz}\right) = S$ et prima integratione per solam variabilem x instituta obtinetur

$$\left(\frac{ddv}{dy\,dz}\right) = \int S dx + f:(y \text{ et } z)$$

nunc secundo integretur per solam variabilem y ac reperietur

$$\left(\frac{dv}{dz}\right) = \iint S\,dx\,dy + \Gamma:(y \text{ et } z) + \Delta:(x \text{ et } z)$$

vnde tandem tertia integratio per z dabit

$$v = \int^{3} S\,dx\,dy\,dz + \Gamma:(y \text{ et } z) + \Delta:(x \text{ et } z) + \Sigma:(x \text{ et } y)$$

sicque problema perfecte est resolutum.

Coroll. 1.

458. Quoniam hic triplici opus erat integratione, integralia inuenta etiam tres functiones arbitrarias complectuntur, easque singulas binarum variabilium, quemadmodum natura integralium completorum postulat.

Coroll. 2.

Coroll. 2.

459. Si quantitas data S euanefcat, integralia haec fequenti modo fe habebunt:

fi fuerit $\left(\frac{d^3 v}{dx^3}\right) = 0$ erit

$$v = xx\Gamma:(y \text{ et } z) + x\Delta:(y \text{ et } z) + \Sigma:(y \text{ et } z)$$

fi fuerit $\left(\frac{d^3 v}{dx^2\, dy}\right) = 0$ erit

$$v = x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z) + \Sigma:(x \text{ et } z)$$

fi fuerit $\left(\frac{d^3 v}{dx\, dy\, dz}\right) = 0$ erit

$$v = \Gamma:(y \text{ et } z) + \Delta:(x \text{ et } z) + \Sigma:(x \text{ et } y).$$

Scholion.

460. Eadem integralia etiam altera methodo fupra expofita inueniri poffunt, fuperfluumque foret fingulas operationes hic apponere. Aeque parum autem opus erit has inueftigationes ad formulas differentiales altiorum graduum profequi, cum lex progreffionis functionum arbitrariarum fingulas integralium partes conftituentium cum per fe tum per ea quae fupra funt expofita, fatis fit manifefta. Quare huic capiti, quo vna quaedam formula differentialis quantitati datae aequari debet, plene eft fatisfactum. Antequam autem vlterius progredior duos adhuc cafus fatis late patentes proponam, quorum refolutio facile ad praecedentes iam tractatas calculi integralis partes reducitur, quam propterea hic

hic tanquam concessam assumere licet, siquidem difficultates, quae in iis occurrunt, non ad praesens institutum sunt referendae.

Problema 77.

461. Si in relationem propositam ex qua naturam functionis trium variabilium x, y et z definiri oportet, aliae formulae differentiales non ingrediantur, nisi quae ex vnica variabili x oriuntur, quae sunt

$$\left(\frac{dv}{dx}\right),\ \left(\frac{ddv}{dx^2}\right),\ \left(\frac{d^3v}{dx^3}\right) \text{ etc.}$$

functionem quaesitam inuestigare.

Solutio.

Cum aequatio propositam continens relationem alias formulas differentiales praeter memoratas non comprehendat, in ea binae quantitates y et z pro constantibus habentur, ideoque etiam in singulis integrationibus tanquam tales tractari possunt. Hinc aequatio proposita duas tantum variabiles x et v inuoluere est censenda, et reiectis formularum differentialium vinculis, habebitur aequatio differentialis ad librum primum referenda in qua, si ad altiores gradus exsurgat, elementum dx constans sumtum est putandum. Quodsi ergo praeceptorum ibidem traditorum ope haec aequatio integrari queat, tum loco constantium per singulas integrationes ingressa-gressarum substituantur functiones arbitrariae binarum variabilium y et z, veluti

$$\Gamma:(y \text{ et } z),\ \Delta:(y \text{ et } z) \text{ etc.}$$

sicque habebitur solutio completa problematis propositi.

Coroll. 1.

462. Praeter plurimos igitur integrabilitatis casus in libro I. expositos, etiam sequentes aequationes differentiales quamtumuis alti gradus resolutionem admittent:

$$S = Av + B\left(\frac{dv}{dx}\right) + C\left(\frac{ddv}{dx^2}\right) + D\left(\frac{d^3v}{dx^3}\right) + \text{etc. et}$$

$$S = Av + Bx\left(\frac{dv}{dx}\right) + Cx^2\left(\frac{ddv}{dx^2}\right) + Dx^3\left(\frac{d^3v}{dx^3}\right) + \text{etc.}$$

Coroll. 2.

463. Vinculis enim abiectis eiusmodi habentur aequationes differentiales, quales in extremis capitibus libri I. integrare docuimus. Tantum opus est, vt loco constantium per integrationes ingressarum scribantur tales functiones:

$$\Gamma:(y \text{ et } z);\ \Delta:(y \text{ et } z);\ \Sigma:(y \text{ et } z) \text{ etc.}$$

vt hoc pacto integralia completa obtineantur.

Scholion.

464. Huc etiam referri possunt eiusmodi relationes propositae, in quibus formulae differentiales

les bina elementa dx et dy inuoluentes ita continentur, vt hoc dy vbique eundem habeat dimenfionum numerum, cuiusmodi funt

$$\left(\frac{dv}{dy}\right);\ \left(\frac{ddv}{dxdy}\right);\ \left(\frac{d^3v}{dx^2dy}\right);\ \left(\frac{d^4v}{dx^3dy}\right)\text{ etc. vel}$$
$$\left(\frac{ddv}{dy^2}\right);\ \left(\frac{d^3v}{dx\,dy^2}\right);\ \left(\frac{d^4v}{dx^2dy^2}\right);\ \left(\frac{d^5v}{dx^3dy^2}\right)\text{ etc.}$$

ipfa autem tum quantitas v nusquam occurrat. Si enim tum pro priori cafu ponatur $\left(\frac{dv}{dy}\right)=u$, pro pofteriori vero $\left(\frac{ddv}{dy^2}\right)=u$, relatio ad cafum problematis reuocabitur, alias formulas differentiales non continens praeter

$$\left(\frac{du}{dx}\right),\ \left(\frac{ddu}{dx^2}\right),\ \left(\frac{d^3u}{dx^3}\right)\text{ etc.}$$

et ipfam forte functionem u Quare fi aequationem per praecepta fupra tradita integrare, in eque functionem u definire licuerit, tum reftituendo loco u vel $\left(\frac{dv}{dy}\right)$ vel $\left(\frac{ddv}{dy^2}\right)$ vt fiat $\left(\frac{dv}{dy}\right)=S$ vel $\left(\frac{ddv}{dy^2}\right)=S$, etiam hinc per praecepta huius capitis ipfa functio v determinabitur. Quin etiam hoc modo refolui poterunt aequationes huiusmodi tantum formulas differentiales complectentes:

$$\left(\frac{d^{\mu+\nu}v}{dy^\mu dz^\nu}\right);\ \left(\frac{d^{\mu+\nu+1}v}{dx\,dy^\mu dz^\nu}\right);\ \left(\frac{d^{\mu+\nu+2}v}{dx^2dy^\mu dz^\nu}\right)\text{ etc.}$$

vbi omnia tria elementa dx, dy, dz occurrunt; pofito enim $\left(\frac{d^{\mu+\nu}v}{dy^\mu dz^\nu}\right)=u$, tota aequatio alias formulas non continebit praeter

$$\left(\frac{du}{dx}\right),\ \left(\frac{ddu}{dx^2}\right),\ \left(\frac{d^3u}{dx^3}\right)\text{ etc.}$$

vna

vna cum ipſa functione u, ſicque ad caſum huius problematis erit referenda ex cuius reſolutione ſi prodierit $u = S = \left(\frac{d^{\mu+\nu} v}{dy^{\mu} dz^{\nu}}\right)$, exiſtente iam S functione cognita, inueſtigatio ipſius functionis v iam nulla amplius laborat difficultate. Datur autem praeterea alius caſus ad libri II. partem priorem redu-

Problema 78.

465. Si in relationem propoſitam, ex qua trium variabilium x, y, z functionem v definiri oportet, aliae formulae differentiales non ingrediuntur, niſi quae ex variabilitate binarum x et y tantum naſcuntur, tertio elemento dz penitus excluſo, functionem v inueſtigare.

Solutio.

Quoniam in aequationem reſoluendam, qua relatio propoſita continetur, quantitas z non vt variabilis ingreditur, quotcunque integrationes fuerint inſtituendae, in iis ita quantitas z tanquam eſſet conſtans tractari debet. Huius ergo aequationis reſolutio ad partem praecedentem eſt referenda, cum functio binarum tantum variabilium x et y ex formularum differentialium relatione data ſit inueſtiganda; quodſi itaque negotium ſucceſſerit et integrale fuerit inuentum, in eo totidem occurrent

 fun-

functiones arbitrariae vnius variabilis certo modo ex x et y conflatae, quot integrationibus fuerit opus; fit $\Gamma:t$ huiusmodi functio, vbi t per x et y dari assumitur: ac nunc vt ista solutio ad praesens institutum accommodetur, vbi quantitas z variabilibus annumeratur, loco cuiusque functionis arbitrariae $\Gamma:t$ scribatur hic $\Gamma:(t \text{ et } z)$ functio scilicet duarum variabilium.

Coroll. 1.

466. Si ergo haec proposita fuerit aequatio

$$\left(\frac{ddv}{dy^2}\right) = aa\left(\frac{ddv}{dx^2}\right)$$

quia in parte praecedente inuenimus

$$v = \Gamma:(x+ay) + \Delta:(x-ay)$$

pro casu praesente, quo v debet esse functio trium variabilium x, y et z integrale ita se habebit:

$$v = \Gamma:(\overline{x+ay} \text{ et } z) + \Delta:(\overline{x-ay} \text{ et } z).$$

Coroll. 2.

467. Hic scilicet meminisse oportet formam

$$\Gamma:(\overline{x+ay} \text{ et } z)$$

designare functionem quamcunque binarum variabilium, quarum altera sit $= x+ay$, altera vero $= z$; vnde ipsam functionem per applicatam ad certam superficiem relatam repraesentare licebit.

Scholion.

Scholion.

468. Non solum autem aequationes in problemate descriptae ad partem praecedentem calculi integralis reducentur, sed etiam innumerabiles aliae, quae facta quadam substitutione ad eam formam reformulae [illegible] si in aequatione proposita aliae omnibus vnica dimensio [illegible] nisi in quibus quae sunt:

$(\frac{dv}{dz})$; $(\frac{ddv}{dxdz})$; $(\frac{ddv}{dydz})$; $(\frac{d^3v}{dx^2dz})$; $(\frac{d^3v}{dxdydz})$; $(\frac{d^3v}{dy^2dz})$ etc

manifestum est posito $(\frac{dv}{dz}) = u$, aequationem illam in aliam transformari, ex qua iam functionem u inuestigari oporteat, eamque ad casum in problemate expositum referri. Quare si inde indoles functionis u definiri potuerit, vt sit $u = S$, restat vt haec aequatio $(\frac{dv}{dz}) = S$ resoluatur, vnde vt ante vidimus, fit

$$v = \int S dz + \Gamma:(x \text{ et } y).$$

Hoc idem tenendum est, si aequatio proposita ope substitutionis

$$(\frac{ddv}{dz^2}) = u \text{ vel } (\frac{d^3v}{dz^3}) = u \text{ etc.}$$

ad casum problematis reduci queat. Quin etiam per se est perspicuum si ope transformationis cuiuscunque, aequatio proposita ad casum problematis reduci queat; tales autem transformationes supra plures exposui, dum vel loco functionis quaesitae v alia

alia u introducitur ponendo $v = Su$, vel ipsae variabiles x, y, z in alias p, q, r mutantur, quae ad illas certam teneant rationem, quod negotium pro casu duarum variabilium supra fusius explicaui; hocque ita perspicuum est, vt similis reductio ad hunc casum trium variabilium facile accommodari queat. In sequentibus tamen forte eiusmodi tra[illegible] generis occurrent; ad alios ergo [illegible] progredior, vix form[illegible] [illegible]menta rem producturus.

CAPVT

CAPVT III.

DE RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS.

Problema 79.

469.

Si pro functione v trium variabilium x, y, z, posito

$$dv = pdx + qdy + rdz$$

fuerit

$$\alpha p + \beta q + \gamma r = 0,$$

indolem functionis v definire.

Solutio.

Cum sit

$$\gamma dv = \gamma pdx + \gamma qdy - (\alpha p + \beta q) dz \text{ erit}$$

$$\gamma dv = p(\gamma dx - \alpha dz) + q(\gamma dy - \beta dz)$$

ideoque ponendo

$$\gamma x - \alpha z = t \text{ et } \gamma y - \beta z = u,$$

habebitur

$$\gamma dv = pdt + qdu$$

vnde

vnde patet quantitatem v aequari functioni cuicunque binarum variabilium t et u, ita vt sit

$$v = \Gamma : (t \text{ et } u)$$

et restitutis valoribus assumtis

$$v = \Gamma : \overline{(\gamma x - \alpha z} \text{ et } \overline{\gamma y - \beta z})$$

quae ergo est solutio problematis, si inter formulas differentiales proponatur haec conditio vt sit

$$\alpha\left(\frac{dv}{dx}\right) + \beta\left(\frac{dv}{dy}\right) + \gamma\left(\frac{dv}{dz}\right) = 0$$

cuius itaque aequationis integrale clarius ita exhibetur:

$$v = \Gamma : \left(\overline{\frac{x}{\alpha} - \frac{z}{\gamma}} \text{ et } \overline{\frac{y}{\beta} - \frac{z}{\gamma}}\right).$$

Coroll. 1.

470. Euidens est hoc integrale etiam ita exprimi posse

$$v = \Gamma : \left(\overline{\frac{x}{\alpha} - \frac{y}{\beta}} \text{ et } \overline{\frac{y}{\beta} - \frac{z}{\gamma}}\right)$$

quandoquidem in genere vti supra obseruauimus est

$$\Gamma : (x \text{ et } y) = \Delta : (t \text{ et } u),$$

siquidem t et u vtcunque per x et y determinentur.

Coroll. 2.

471. Quin etiam affirmare licet, constitutis his tribus formulis

$$\frac{x}{\alpha} - \frac{y}{\beta}; \quad \frac{y}{\beta} - \frac{z}{\gamma}; \quad \frac{z}{\gamma} - \frac{x}{\alpha}$$

quantitatem v esse functionem quamcunque trium harum

harum formularum; ſiquidem vnaquaeque iam per binas reliquas datur, ac propterea v nihilominus functioni duarum tantum quantitatum variabilium aequatur.

Problema 80.

472. Si poſito

$$dv = p\,dx + q\,dy + r\,dz$$

haec conditio requiratur vt ſit

$$px + qy + rz = nv \text{ ſeu}$$

$$nv = x\left(\frac{dv}{dx}\right) + y\left(\frac{dv}{dy}\right) + z\left(\frac{dv}{dz}\right)$$

indolem huius functionis v inueſtigare.

Solutio.

Ex conditione praeſcripta capiatur valor $r = \frac{nv - px - qy}{z}$ quo ſubſtituto fit

$$dv - \frac{nv\,dz}{z} = p\left(dx - \frac{x\,dz}{z}\right) + q\left(dy - \frac{y\,dz}{z}\right) \text{ ſeu}$$

$$dv - \frac{nv\,dz}{z} = pz\,d.\frac{x}{z} + qz\,d.\frac{y}{z}.$$

Quo primum membrum integrabile reddatur multiplicetur per $\frac{1}{z^n}$, ita vt iam habeamus:

$$d.\frac{v}{z^n} = \frac{pz}{z^n}\,d.\frac{x}{z} + \frac{qz}{z^n}\,d.\frac{y}{z}.$$

Cum nunc quantitates p et q non ſint determinatae sufficit hanc vnam $\left(\frac{ddv}{dx\,dy}\right)=S$ resoluisse; quae primo per solam variabilem x integrata dabit

$$\left(\frac{dv}{dy}\right)=\int S\,dx+f:(y \text{ et } z).$$

Deinde altera integratione per solam variabilem y instituta colligitur:

$$v=\int dy\int S\,dx+\int dy\,f:(y \text{ et } z)+\Delta:(x \text{ et } z)$$

vbi primum obseruo partem primam nullo discrimine ordinis inter binas variabiles x et y habito ita $\int\int S\,dx\,dy$ exprimi posse. Deinde quaecunque fuerit $f:(y \text{ et } z)$ functio ipsarum y et z, si ea per dy multiplicetur et spectata z vt constante integretur, euidens est denuo functionem ipsarum y et z prodire, et quia illa nullo modo determinatur, etiam hanc fore indeterminatam ideoque arbitrariam, vnde statuere poterimus:

$$v=\int\int S\,dx\,dy+\Gamma:(y \text{ et } z)+\Delta:(x \text{ et } z).$$

Coroll. 1.

453. Hic obseruo per integrationem formulae $\int dy\,f:(y \text{ et } z)$ iam sponte formulam $\Delta:(x \text{ et } z)$ inuehi; cum enim ibi sola quantitas y vt variabilis spectetur, loco quantitatis constantis per integrationem adiiciendae functio quaecunque ipsarum x et z scribi poterit.

Coroll. 2.

Coroll. 2.

454. Quodsi functio illa data S euanescat, sequentes integrationes prouenient:

si $\left(\frac{ddv}{dx^2}\right)=0$, erit $v=x\Gamma:(y \text{ et } z)+\Delta:(y \text{ et } z)$

si $\left(\frac{ddv}{dy^2}\right)=0$, erit $v=y\Gamma:(x \text{ et } z)+\Delta:(x \text{ et } z)$

si $\left(\frac{ddv}{dz^2}\right)=0$, erit $v=z\Gamma:(x \text{ et } y)+\Delta:(x \text{ et } y)$

si $\left(\frac{ddv}{dx\,dy}\right)=0$, erit $v=\Gamma:(x \text{ et } z)+\Delta:(y \text{ et } z)$

si $\left(\frac{ddv}{dx\,dz}\right)=0$, erit $v=\Gamma:(x \text{ et } y)+\Delta:(y \text{ et } z)$

si $\left(\frac{ddv}{dy\,dz}\right)=0$, erit $v=\Gamma:(x \text{ et } y)+\Delta:(x \text{ et } z)$.

Coroll. 3.

455. Quia hic duplici opus est integratione, atque etiam duae functiones arbitrariae, vtraque binarum variabilium in calculum sunt inuectae; hoc certissimum est criterium haec integralia inuenta esse completa.

Scholion.

456. Alio etiam modo haec eadem integralia erui possunt, qui nititur principio supra (451.) indicato, quod si fuerit

$dv=S\,dx+q\,dy+r\,dz$ fore

$v=\int S\,dx+f:(y \text{ et } z)$.

Secundum hoc principium ergo si fuerit $\left(\frac{ddv}{dx^2}\right)=S$ erit

$$d.\left(\frac{dv}{dx}\right)=S\,dx+dy\left(\frac{ddv}{dx\,dy}\right)+dz\left(\frac{ddv}{dx\,dz}\right),$$

qua forma cum illa collata loco v habemus $\left(\frac{dv}{dx}\right)$ et loco q et r has formulas

$$\left(\frac{ddv}{dx\,dy}\right) \text{ et } \left(\frac{ddv}{dx\,dz}\right),$$

ex quo integrale erit

$$\left(\frac{dv}{dx}\right)=\int S\,dx+f:(y \text{ et } z).$$

Cum iam porro sit

$$dv=\left(\frac{dv}{dx}\right)dx+\left(\frac{dv}{dy}\right)dy+\left(\frac{dv}{dz}\right)dz \text{ erit}$$

$$dv=dx\int S\,dx+dx\,f:(y \text{ et } z)+dy\left(\frac{dv}{dy}\right)+dz\left(\frac{dv}{dz}\right)$$

vnde pariter manifesto sequitur:

$$v=\int dx\int S\,dx+x\,\Gamma:(y \text{ et } z)+\Delta:(y \text{ et } z).$$

Pari modo operatio est instituenda pro aequatione $\left(\frac{ddv}{dx\,dy}\right)=S$, inde enim fit

$$d.\left(\frac{dv}{dy}\right)=S\,dx+dy\left(\frac{ddv}{dy^2}\right)+dz\left(\frac{ddv}{dy\,dz}\right)$$

cuius integrale est

$$\left(\frac{dv}{dy}\right)=\int S\,dx+f:(y \text{ et } z);$$

altera integratio instituatur in hac forma

$$dv=dy\int S\,dx+dy\,f:(y \text{ et } z)+dx\left(\frac{dv}{dx}\right)+dz\left(\frac{dv}{dz}\right)$$

vnde ob

$$\int dy\,f:(y \text{ et } z)=\Gamma:(y \text{ et } z)$$

obti-

obtinetur vt ante:

$$v = \iint S\,dx\,dy + \Gamma:(y \text{ et } z) + \Delta:(x \text{ et } z).$$

Problema 76.

457. Inueſtigare indolem functionis trium variabilium x, y et z, cuius quaedam formula differentialis tertii gradus aequetur datae cuipiam quantitati S ex illis variabilibus et conſtantibus vtcunque compoſitae.

Solutio.

Poſita functione quaeſita $= v$, percurramus non tam ſingulas eius formulas differentiales tertii gradus, quam eas quarum ratio eſt diuerſa.

Sit igitur primo $\left(\frac{d^3 v}{dx^3}\right) = S$, et prima integratio ſtatim dat

$$\left(\frac{ddv}{dx^2}\right) = \int S\,dx + 2\Gamma:(y \text{ et } z),$$

tum vero altera

$$\left(\frac{dv}{dx}\right) = \int dx \int S\,dx + 2x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z)$$

vnde tandem colligitur:

$$v = \int dx \int dx \int S\,dx + xx\Gamma:(y \text{ et } z) + x\Delta:(y \text{ et } z) + \Sigma:(y \text{ et } z).$$

Sit ſecundo $\left(\frac{d^3 v}{dx^2\,dy}\right) = S$ et binae priores integrationes vt ante dant:

$$\left(\frac{dv}{dy}\right) = \int dx \int S\,dx + x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z)$$

 quia

quia nunc vt vidimus pro $\int dy\Gamma:(y \text{ et } z)$ ſcribere licet: $\Gamma:(y \text{ et } z)$ per tertiam integrationem inuenimus:

$$v = \int^2 S dx^2 dy + x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z) + \Sigma:(x \text{ et } z).$$

In his autem duobus caſibus omnes formulae differentiales tertii gradus, variabilibus permutandis, continentur, ſola excepta vltima hac $\left(\frac{d^3 v}{dx\,dy\,dz}\right)$, quam idcirco ſeorſim tractari oportet:

Sit igitur $\left(\frac{d^3 v}{dx\,dy\,dz}\right) = S$ et prima integratione per ſolam variabilem x inſtituta obtinetur

$$\left(\frac{ddv}{dy\,dz}\right) = \int S dx + f:(y \text{ et } z)$$

nunc ſecundo integretur per ſolam variabilem y, ac reperietur

$$\left(\frac{dv}{dz}\right) = \int\int S\,dx\,dy + \Gamma:(y \text{ et } z) + \Delta:(x \text{ et } z)$$

vnde tandem tertia integratio per z dabit

$$v = \int^3 S\,dx\,dy\,dz + \Gamma:(y \text{ et } z) + \Delta:(x \text{ et } z) + \Sigma:(x \text{ et } y)$$

ſicque problema perfecte eſt reſolutum.

Coroll. 1.

458. Quoniam hic triplici opus erat integratione, integralia inuenta etiam tres functiones arbitrarias complectuntur, easque ſingulas binarum variabilium, quemadmodum natura integralium completorum poſtulat.

Coroll. 2.

459. Si quantitas data S euanescat, integralia haec sequenti modo se habebunt:

si fuerit $\left(\frac{d^3 v}{dx^3}\right) = 0$ erit

$$v = xx\Gamma:(y \text{ et } z) + x\Delta:(y \text{ et } z) + \Sigma:(y \text{ et } z)$$

si fuerit $\left(\frac{d^3 v}{dx^2\, dy}\right) = 0$ erit

$$v = x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z) + \Sigma:(x \text{ et } z)$$

si fuerit $\left(\frac{d^3 v}{dx\, dy\, dz}\right) = 0$ erit

$$v = \Gamma:(y \text{ et } z) + \Delta:(x \text{ et } z) + \Sigma:(x \text{ et } y).$$

Scholion.

460. Eadem integralia etiam altera methodo supra exposita inueniri possunt, superfluumque foret singulas operationes hic apponere. Aeque parum autem opus erit has inuestigationes ad formulas differentiales altiorum graduum prosequi, cum lex progressionis functionum arbitrariarum singulas integralium partes constituentium cum per se tum per ea quae supra sunt exposita, satis sit manifesta. Quare huic capiti, quo vna quaedam formula differentialis quantitati datae aequari debet, plene est satisfactum. Antequam autem vlterius progredior duos adhuc casus satis late patentes proponam, quorum resolutio facile ad praecedentes iam tractatas calculi integralis partes reducitur, quam propterea hic

hic tanquam concessam assumere licet, siquidem difficultates, quae in iis occurrunt, non ad praesens institutum sunt referendae.

Problema 77.

461. Si in relationem propositam ex qua naturam functionis trium variabilium x, y et z definiri oportet, aliae formulae differentiales non ingrediantur, nisi quae ex vnica variabili x oriuntur, quae sunt

$$\left(\frac{dv}{dx}\right),\ \left(\frac{ddv}{dx^2}\right),\ \left(\frac{d^3v}{dx^3}\right) \text{ etc.}$$

functionem quaesitam inuestigare.

Solutio.

Cum aequatio propositam continens relationem alias formulas differentiales praeter memoratas non comprehendat, in ea binae quantitates y et z pro constantibus habentur, ideoque etiam in singulis integrationibus tanquam tales tractari possunt. Hinc aequatio proposita duas tantum variabiles x et v inuoluere est censenda, et reiectis formularum differentialium vinculis, habebitur aequatio differentialis ad librum primum referenda in qua, si ad altiores gradus exsurgat, elementum dx constans sumtum est putandum. Quodsi ergo praeceptorum ibidem traditorum ope haec aequatio integrari queat, tum loco constantium per singulas integrationes ingressa-greſſarum ſubſtituantur functiones arbitrariae binarum variabilium y et z, veluti

$$\Gamma:(y \text{ et } z),\ \Delta:(y \text{ et } z) \text{ etc.}$$

ſicque habebitur ſolutio completa problematis propoſiti.

Coroll. 1.

462. Praeter plurimos igitur integrabilitatis caſus in libro I. expoſitos, etiam ſequentes aequationes differentiales quamtumuis alti gradus reſolutionem admittent:

$$S = Av + B\left(\frac{dv}{dx}\right) + C\left(\frac{ddv}{dx^2}\right) + D\left(\frac{d^3v}{dx^3}\right) + \text{etc. et}$$

$$S = Av + Bx\left(\frac{dv}{dx}\right) + Cx^2\left(\frac{ddv}{dx^2}\right) + Dx^3\left(\frac{d^3v}{dx^3}\right) + \text{etc.}$$

Coroll. 2.

463. Vinculis enim abiectis eiusmodi habentur aequationes differentiales, quales in extremis capitibus libri I. integrare docuimus. Tantum opus eſt, vt loco conſtantium per integrationes ingreſſarum ſcribantur tales functiones:

$$\Gamma:(y \text{ et } z);\ \Delta:(y \text{ et } z);\ \Sigma:(y \text{ et } z) \text{ etc.}$$

vt hoc pacto integralia completa obtineantur.

Scholion.

464. Huc etiam referri poſſunt eiusmodi relationes propoſitae, in quibus formulae differentia-

les bina elementa dx et dy inuoluentes ita continentur, vt hoc dy vbique eundem habeat dimensionum numerum, cuiusmodi sunt

$$\left(\frac{dv}{dy}\right);\ \left(\frac{ddv}{dx\,dy}\right);\ \left(\frac{d^3v}{dx^2\,dy}\right);\ \left(\frac{d^4v}{dx^3\,dy}\right) \text{ etc. vel}$$
$$\left(\frac{ddv}{dy^2}\right);\ \left(\frac{d^3v}{dx\,dy^2}\right);\ \left(\frac{d^4v}{dx^2\,dy^2}\right);\ \left(\frac{d^5v}{dx^3\,dy^2}\right) \text{ etc.}$$

ipsa autem tum quantitas v nusquam occurrat. Si enim tum pro priori casu ponatur $\left(\frac{dv}{dy}\right)=u$, pro posteriori vero $\left(\frac{ddv}{dy^2}\right)=u$, relatio ad casum problematis reuocabitur, alias formulas differentiales non continens praeter

$$\left(\frac{du}{dx}\right),\ \left(\frac{ddu}{dx^2}\right),\ \left(\frac{d^3u}{dx^3}\right) \text{ etc.}$$

et ipsam forte functionem u Quare si aequationem per praecepta supra tradita integrare, in eque functionem u definire licuerit, tum restituendo loco u vel $\left(\frac{dv}{dy}\right)$ vel $\left(\frac{ddv}{dy^2}\right)$ vt fiat $\left(\frac{dv}{dy}\right)=S$ vel $\left(\frac{ddv}{dy^2}\right)=S$, etiam hinc per praecepta huius capitis ipsa functio v determinabitur. Quin etiam hoc modo resolui poterunt aequationes huiusmodi tantum formulas differentiales complectentes:

$$\left(\frac{d^{\mu+\nu}v}{dy^\mu\,dz^\nu}\right);\ \left(\frac{d^{\mu+\nu+1}v}{dx\,dy^\mu\,dz^\nu}\right);\ \left(\frac{d^{\mu+\nu+2}v}{dx^2\,dy^\mu\,dz^\nu}\right) \text{ etc.}$$

vbi omnia tria elementa dx, dy, dz occurrunt; posito enim $\left(\frac{d^{\mu+\nu}v}{dy^\mu\,dz^\nu}\right)=u$, tota aequatio alias formulas non continebit praeter

$$\left(\frac{du}{dx}\right),\ \left(\frac{ddu}{dx^2}\right),\ \left(\frac{d^3u}{dx^3}\right) \text{ etc.}$$

vna

vna cum ipsa functione u, sicque ad casum huius problematis erit referenda ex cuius resolutione si prodierit $u = S = \left(\frac{d^{\mu+\nu}v}{dy^{\mu}dz^{\nu}}\right)$, existente iam S functione cognita, inuestigatio ipsius functionis v iam nulla amplius laborat difficultate. Datur autem praeterea alius casus ad libri II. partem priorem redu- [illegible]

Problema 78.

465. Si in relationem propositam, ex qua trium variabilium x, y, z functionem v definiri oportet, aliae formulae differentiales non ingrediuntur, nisi quae ex variabilitate binarum x et y tantum nascuntur, tertio elemento dz penitus excluso, functionem v inuestigare.

Solutio.

Quoniam in aequationem resoluendam, qua relatio proposita continetur, quantitas z non vt variabilis ingreditur, quotcunque integrationes fuerint instituendae, in iis ita quantitas z tanquam esset constans tractari debet. Huius ergo aequationis resolutio ad partem praecedentem est referenda, cum functio binarum tantum variabilium x et y ex formularum differentialium relatione data sit inuestiganda; quodsi itaque negotium successerit et integrale fuerit inuentum, in eo totidem occurrent functiones arbitrariae vnius variabilis certo modo ex x et y conflatae, quot integrationibus fuerit opus; sit $\Gamma : t$ huiusmodi functio, vbi t per x et y dari assumitur: ac nunc vt ista solutio ad praesens institutum accommodetur, vbi quantitas z variabilibus annumeratur, loco cuiusque functionis arbitrariae $\Gamma : t$ scribatur hic $\Gamma : (t \text{ et } z)$ functio scilicet duarum variabilium [illegible]

Coroll. 1.

466. Si ergo haec proposita fuerit aequatio

$$\left(\frac{ddv}{dy^2}\right) = aa\left(\frac{ddv}{dx^2}\right)$$

quia in parte praecedente inuenimus

$$v = \Gamma : (x + ay) + \Delta : (x - ay)$$

pro casu praesente, quo v debet esse functio trium variabilium x, y et z integrale ita se habebit:

$$v = \Gamma : (\overline{x + ay} \text{ et } z) + \Delta : (\overline{x - ay} \text{ et } z).$$

Coroll. 2.

467. Hic scilicet meminisse oportet formam

$$\Gamma : (\overline{x + ay} \text{ et } z)$$

designare functionem quamcunque binarum variabilium, quarum altera sit $= x + ay$, altera vero $= z$; vnde ipsam functionem per applicatam ad certam superficiem relatam repraesentare licebit.

Scholion.

Scholion.

468. Non ſolum autem aequationes in problemate deſcriptae ad partem praecedentem calculi integralis reducentur, ſed etiam innumerabiles aliae, quae facta quadam ſubſtitutione ad eam formam reformulae [illegible] veluti ſi in aequatione propoſita aliae omnibus vnica dimenſio [illegible] niſi in quibus quae ſunt:

$\left(\frac{dv}{dz}\right); \left(\frac{ddv}{dxdz}\right); \left(\frac{ddv}{dydz}\right); \left(\frac{d^3v}{dx^2dz}\right); \left(\frac{d^3v}{dxdydz}\right); \left(\frac{d^3v}{dy^2dz}\right)$ etc

manifeſtum eſt poſito $\left(\frac{dv}{dz}\right) = u$, aequationem illam in aliam transformari, ex qua iam functionem u inueſtigari oporteat, eamque ad caſum in problemate expoſitum referri. Quare ſi inde indoles functionis u definiri potuerit, vt ſit $u = S$, reſtat vt haec aequatio $\left(\frac{dv}{dz}\right) = S$ reſoluatur, vnde vt ante vidimus, fit

$$v = \int S dz + \Gamma:(x \text{ et } y).$$

Hoc idem tenendum eſt, ſi aequatio propoſita ope ſubſtitutionis

$$\left(\frac{ddv}{dz^2}\right) = u \quad \text{vel} \quad \left(\frac{d^3v}{dz^3}\right) = u \text{ etc.}$$

ad caſum problematis reduci queat. Quin etiam per ſe eſt perſpicuum ſi ope transformationis cuiuscunque, aequatio propoſita ad caſum problematis reduci queat; tales autem transformationes ſupra plures expoſui, dum vel loco functionis quaeſitae v

 alia

alia u introducitur ponendo $v = Su$, vel ipſae variabiles x, y, z in alias p, q, r mutantur, quae ad illas certam teneant rationem, quod negotium pro caſu duarum variabilium ſupra fuſius explicaui; hocque ita perſpicuum eſt, vt ſimilis reductio ad hunc caſum trium variabilium facile accommodari queat. In ſequentibus tamen forte eiusmodi tra[illegible] generis occurrent; ad alios ergo [illegible] progredior, vix form[illegible] elementa rem producturus.

CAPVT

CAPVT III.

DE RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS.

Problema 79.

469.

Si pro functione v trium variabilium x, y, z, posito

$$dv = pdx + qdy + rdz$$

fuerit

$$\alpha p + \beta q + \gamma r = 0,$$

indolem functionis v definire.

Solutio.

Cum fit

$$\gamma dv = \gamma pdx + \gamma qdy - (\alpha p + \beta q) dz \text{ erit}$$

$$\gamma dv = p(\gamma dx - \alpha dz) + q(\gamma dy - \beta dz)$$

ideoque ponendo

$$\gamma x - \alpha z = t \text{ et } \gamma y - \beta z = u,$$

habebitur

$$\gamma dv = pdt + qdu$$

vnde

vnde patet quantitatem v aequari functioni cuicunque binarum variabilium t et u, ita vt sit

$$v = \Gamma : (t \text{ et } u)$$

et restitutis valoribus assumtis

$$v = \Gamma : (\overline{\gamma x - \alpha z} \text{ et } \overline{\gamma y - \beta z}).$$

quae ergo est solutio problematis, si inter formulas differentiales proponatur haec conditio vt sit

$$\alpha\left(\frac{dv}{dx}\right) + \beta\left(\frac{dv}{dy}\right) + \gamma\left(\frac{dv}{dz}\right) = 0$$

cuius itaque aequationis integrale clarius ita exhibetur:

$$v = \Gamma : (\overline{\tfrac{x}{\alpha} - \tfrac{z}{\gamma}} \text{ et } \overline{\tfrac{y}{\beta} - \tfrac{z}{\gamma}}).$$

Coroll. 1.

470. Euidens est hoc integrale etiam ita exprimi posse

$$v = \Gamma : (\overline{\tfrac{x}{\alpha} - \tfrac{y}{\beta}} \text{ et } \overline{\tfrac{y}{\beta} - \tfrac{z}{\gamma}})$$

quandoquidem in genere vti supra obseruauimus est

$$\Gamma : (x \text{ et } y) = \Delta : (t \text{ et } u),$$

siquidem t et u vtcunque per x et y determinentur.

Coroll. 2.

471. Quin etiam affirmare licet, constitutis his tribus formulis

$$\frac{x}{\alpha} - \frac{y}{\beta}; \quad \frac{y}{\beta} - \frac{z}{\gamma}; \quad \frac{z}{\gamma} - \frac{x}{\alpha}$$

quantitatem v esse functionem quamcunque trium harum

harum formularum; siquidem vnaquaeque iam per binas reliquas datur, ac propterea v nihilominus functioni duarum tantum quantitatum variabilium aequatur.

Problema 80.

472. Si posito

$$dv = p\,dx + q\,dy + r\,dz$$

haec conditio requiratur vt sit

$$px + qy + rz = nv \text{ seu}$$

$$nv = x\left(\frac{dv}{dx}\right) + y\left(\frac{dv}{dy}\right) + z\left(\frac{dv}{dz}\right)$$

indolem huius functionis v inuestigare.

Solutio.

Ex conditione praescripta capiatur valor $r = \frac{nv - px - qy}{z}$ quo substituto fit

$$dv - \frac{nv\,dz}{z} = p\left(dx - \frac{x\,dz}{z}\right) + q\left(dy - \frac{y\,dz}{z}\right) \text{ seu}$$

$$dv - \frac{nv\,dz}{z} = pz\,d.\frac{x}{z} + qz\,d.\frac{y}{z}.$$

Quo primum membrum integrabile reddatur multiplicetur per $\frac{1}{z^n}$, ita vt iam habeamus:

$$d.\frac{v}{z^n} = \frac{pz}{z^n}\,d.\frac{x}{z} + \frac{qz}{z^n}\,d.\frac{y}{z}.$$

Cum nunc quantitates p et q non sint determinatae quoniam in genere ex tali aequatione

$$dV = PdX + QdY$$

sequitur

$$V = \Gamma : (X \text{ et } Y),$$

pro nostro casu colligimus:

$$\frac{v}{z^n} = \Gamma : \left(\frac{x}{z} \text{ et } \frac{y}{z}\right) \text{ seu}$$

$$v = z^n \Gamma : \left(\frac{x}{z} \text{ et } \frac{y}{z}\right).$$

Si scilicet functio quaecunque binarum quantitatum $\frac{x}{z}$ et $\frac{y}{z}$ per z^n seu etiam quod eodem redit per x^n vel y^n multiplicetur oritur valor idoneus pro functione v conditioni praescriptae satisfaciens.

Coroll. 1.

473. Perspicuum autem est formam $\Gamma : (\frac{x}{z} \text{ et } \frac{y}{z})$ exprimere eiusmodi functionem in qua tres variabiles x, y, z vbique constituant nullum dimensionum numerum, ac vicissim omnes huiusmodi functiones in forma illa contineri.

Coroll. 2.

474. Multiplicatione autem porro facta per z^n, oritur functio homogenea trium variabilium x, y, z, cuius dimensionum numerus est $= n$; vnde solutio nostri problematis ita enunciari potest, vt quantitas

quae-

quaesita v sit functio homogenea trium variabilium x, y et z dimensionum numero existente $= n$.

Coroll. 3.

475. Quodsi ergo conditio praescripta sit

$$px + qy + rz = 0 \text{ seu}$$

$$x\left(\frac{dv}{dx}\right) + y\left(\frac{dv}{dy}\right) + z\left(\frac{dv}{dz}\right) = 0,$$

quantitas v erit functio homogenea nullius dimensionis trium variabilium x, y et z.

Scholion.

476. Simili modo solutio succedit, si conditio praescripta postulet vt sit

$$\alpha px + \beta qy + \gamma rz = nv \text{ seu}$$

$$\alpha x\left(\frac{dv}{dx}\right) + \beta y\left(\frac{dv}{dy}\right) + \gamma z\left(\frac{dv}{dz}\right) = nv$$

tum enim ob

$$r = \frac{nv - \alpha px - \beta qy}{\gamma z} \text{ fit}$$

$$dv - \frac{nv\,dz}{\gamma z} = p\left(dx - \frac{\alpha x\,dz}{\gamma z}\right) + q\left(dy - \frac{\beta y\,dz}{\gamma z}\right)$$

quae aequatio sequenti forma exhibeatur:

$$\frac{\gamma dv}{v} - \frac{ndz}{z} = \frac{px}{v}\left(\frac{\gamma dx}{x} - \frac{\alpha dz}{z}\right) + \frac{qy}{v}\left(\frac{\gamma dy}{y} - \frac{\beta dz}{z}\right)$$

ex qua concludimus integrale primi membri $\gamma lv - nlz$ aequari functioni cuicunque binarum quantitatum

$$\gamma lx - \alpha lz \text{ et } \gamma ly - \beta lz,$$

 et

et logarithmorum numeris sumtis fore

$$\frac{v^\gamma}{z^n}=\Gamma:\left(\frac{x^\gamma}{z^\alpha}\text{ et }\frac{y^\gamma}{z^\beta}\right).$$

Ponamus $\alpha=\frac{1}{\lambda}$, $\beta=\frac{1}{\mu}$ et $\gamma=\frac{1}{\nu}$; vt conditio praescripta sit

$$\frac{px}{\lambda}+\frac{qy}{\mu}+\frac{rz}{\nu}=nv$$

et solutio reducetur ad hanc formam:

$$v=z^{\nu n}\Delta:\left(\frac{x^\lambda}{z^\nu}\text{ et }\frac{y^\mu}{z^\nu}\right).$$

Quodsi porro scribamus

$$x^\lambda=X,\ y^\mu=Y \text{ et } z^\nu=Z \text{ fiet}$$
$$v=Z^n\Delta:\left(\tfrac{X}{Z}\text{ et }\tfrac{Y}{Z}\right),$$

ideoque quantitas quaesita v est functio homogenea, in qua tres variabiles X, Y et Z vbique eundem dimensionum numerum $=n$ adimplent.

Problema 81.

477. Si posito

$$dv=pdx+qdy+rdz$$

haec conditio praescribatur vt sit

$$px+qy+rz=nv+S,$$

existente S functione quacunque data variabilium x, y, z inuestigare naturam functionis quaesitae v.

Solutio.

Solutio.

Cum conditio praescripta praebeat

$r = \frac{nv+S-px-qy}{z}$ erit

$dv - \frac{nvdz}{z} = \frac{Sdz}{z} + p(dx - \frac{xdz}{z}) + q(dy - \frac{ydz}{z})$ seu

$$d.\frac{v}{z^n} = \frac{Sdz}{z^{n+1}} + \frac{p}{z^{n-1}} d.\frac{x}{z} + \frac{q}{z^{n-1}} d.\frac{y}{z}.$$

Sit $x = tz$ et $y = uz$, vt iam S fiat functio trium variabilium t, u et z, et formula differentialis: $\frac{Sdz}{z^{n+1}}$ ita integretur, vt quantitates t et u constantes habeantur, quo integrali posito $= V$ erit

$$v = Vz^n + z^n\Gamma : (\tfrac{x}{z} \text{ et } \tfrac{y}{z})$$

vbi pars posterior significat functionem homogeneam trium variabilium x, y, z numero dimensionum existente $= n$.

Coroll. 1.

478. Si S sit quantitas constans $= C$ erit

$$V = \int \frac{Cdz}{z^{n+1}} = -\frac{C}{nz^n},$$

hincque primum integralis membrum:

$$Vz^n = -\frac{C}{n};$$

ex quo perspicuum est eundem valorem prodituruin fuisse, quantitatibus x, y, z inter se permutatis.

Coroll. 2.

479. Si S sit functio homogenea ipsarum x, y, z dimensionum numero existente $=m$, quia tum posito $x=tz$ et $y=uz$ fit $S=Mz^m$, ita vt M tantum quantitates t et u inuoluat, ideoque pro constante sit habenda: prodit

$$V=\int Mz^{m-n-1}dz=\frac{Mz^{m-n}}{m-n}=\frac{S}{(m-n)z^n}$$

sicque primum integralis membrum erit $=\frac{S}{m-n}$.

Coroll. 3.

480. At si hoc casu sit $m=n$, fit

$$V=Mlz+C=Mlaz$$

et primum integralis membrum

$$=Mz^n laz=Slaz.$$

Pari iure id autem erit

$$=Slby \text{ vel } Slax,$$

id quod satis est manifestum cum horum valorum differentia fiat functio homogenea n dimensionum, ideoque in altero integralis membro contineatur.

Scholion.

481. Principium huius solutionis in hoc lemmate latissime patente continetur, quod si fuerit

$$dV=SdZ+PdX+QdY$$

vbi

vbi S denotat functionem datam, P et Q vero functiones indefinitas, futurum fit

$$V = \int S dZ + \Gamma : (X \text{ et } Y)$$

at hic non fufficit indicaffe in integratione formulae SdZ folam quantitatem Z pro variabili haberi, fed infuper notari conuenit binas X et Y tanquam conftantes tractari debere. Quare fi forte S fit propofita functio aliarum trium variabilium x, y, z, ex quibus hae X, Y, Z, quarum ratio hic eft habenda, certo modo nafcantur, primum loco x, y, z iftae X, Y et Z introduci debent, vt fiat S functio harum X, Y et Z; tum vero demum binis X et Y pro conftantibus folaque Z pro variabili fumta integrale $\int SdZ$ eft capiendum. Ita in cafu problematis pro integrali $\int \frac{Sdz}{z^{n+1}}$, quantitates $\frac{x}{z}$ et $\frac{y}{z}$ vt conftantes funt fpectandae, fola z pro variabili fumta; ex quo in functione S ftatui oportet $x = tz$ et $y = uz$, vt S fiat functio ipfarum z, $t = \frac{x}{z}$ et $u = \frac{y}{z}$, quarum binae pofteriores pro conftantibus funt habendae. Hoc ergo cafu infignis error committeretur, fi quis fumta z variabili reliquas x et y vt conftantes tractare voluerit, quoniam ambae x et y etiam variabilem z inuoluere funt cenfendae. Quod autem variabilibus permutatis primum integralis membrum idem refultare debeat, vt fit

$$z^n \int \frac{Sdz}{z^{n+1}} = x^n \int \frac{Sdx}{x^{n+1}}$$

Inde

vt fiat

$$dv = pL\,dt + qM\,du,$$

et manifestum est quantitatem v aequari debere functioni cuicunque binarum variabilium t et u, quas ita quoque describere licet, vt positis formulis tribus integralibus $\int\frac{dx}{L}$; $\int\frac{dy}{M}$; et $\int\frac{dz}{N}$; pro t et u sumi oporteat differentias inter binas earum.

Scholion 1.

483. Solutio etiam successisset, dummodo $\frac{L}{N}$ fuisset functio ipsarum x et z, et $\frac{M}{N}$ ipsarum y et z tantum; tum enim multiplicatores P et Q ad integrationem apti quaeri debuissent vt fieret

$$P\left(dx - \frac{L\,dz}{N}\right) = dt \text{ et } Q\left(dy - \frac{M\,dz}{N}\right) = du$$

et ob

$$dv = \frac{p\,dt}{P} + \frac{q\,du}{Q} \text{ foret}$$

$$v = \Gamma:(t \text{ et } u).$$

Permutandis vero variabilibus x, y et z etiam alii casus resolubiles prodeunt. Quando autem quantitates L, M, N aliter sunt comparatae, via non patet certa ad solutionem perueniendi, quae certe haud parum abstrusa videtur, cum pro hoc casu satis simplici

$$(y+z)p + (x+z)q + (x+y)z = 0$$

 per

per plures ambages tandem ad hanc peruenerim solutionem vt posito

$$t=(x+y+z)(x-z)^2 \text{ et } u=(x+y+z)(y-z)^2$$

fiat $v=\Gamma:(t \text{ et } u)$;

quoniam igitur binae quantitates t et u, quarum functio quaecunque loco v posita conditioni satisfacit, hoc casu tantopere sunt complicatae generaliter multo minus solutionem expectare licebit.

Scholion 2.

484. Ad plures autem alios casus solutio extendi potest. Si functiones datae L, M, N ita fuerint comparatae, vt alias E, F, G, H reperire liceat, quibus fiat:

$$E(dx-\tfrac{Ldz}{N})+F(dy-\tfrac{Mdz}{N})=dt \text{ et}$$
$$G(dx-\tfrac{Ldz}{N})+H(dy-\tfrac{Mdz}{N})=du$$

tum enim posito

$$p=PE+QG \text{ et } q=PF+QH, \text{ fiet}$$
$$dv=Pdt+Qdu,$$

vbi P et Q sunt functiones indefinitae loco p et q introductae, quantitas v aequabitur functioni cuicunque binarum variabilium t et u seu erit

$$v=\Gamma:(t \text{ et } u).$$

Totum ergo negotium huc redit, vt pro datis functionibus L, M, N functiones E, F et G, H inueniantur, quod quidem semper praestari posse videtur, sed haec ipsa quae-

quaestio plerumque difficilior euadit quam ipsa proposita. Sufficit autem binas eiusmodi functiones E et F indeque quantitatem t inuestigasse; quia deinceps permutandis variabilibus x, y, z vna cum respondentibus functionibus L, M, N sponte idoneus valor pro u elicitur. Ita in exemplo ante allato

$$L = y + z, \quad M = x + z, \quad N = x + y$$

postquam inuenerimus

$$t = (x + y + z)(x - z)^2,$$

sola permutatio statim praebet

$$u = (x + y + z)(y - z)^2$$

vel etiam

$$u = (x + y + z)(x - y)^2,$$

perinde enim est, vtro valore vtamur.

Problema 83.

485. Si posito

$$dv = p\,dx + q\,dy + r\,dz$$

haec conditio praescribatur vt sit $pqr = 1$, naturam functionis v inuestigare.

Solutio.

Ob $r = \frac{1}{pq}$ erit

$$dv = p\,dx + q\,dy + \frac{dz}{pq},$$

 vnde

vnde colligimus

$$v = px + qy + \frac{z}{pq} - \int\left(x\,dp + y\,dq - \frac{z\,dp}{ppq} - \frac{z\,dq}{pqq}\right)$$

qua transformatione id sumus assecuti, vt formula integralis bina tantum differentialia dp et dq inuoluat. His igitur in locum principalium inductis, concludimus illam formulam integralem aequari debere functioni cuicunque binarum variabilium p et q. Sit S talis functio, vt fiat

$$v = px + qy + \frac{z}{pq} - S,$$

et iam superest, vt cum litterae p et q in calculo retineantur aliae duae elidentur, id quod inde est petendum, quod sit:

$$dS = \left(x - \frac{z}{ppq}\right)dp + \left(y - \frac{z}{pqq}\right)dq$$

ideoque

$$x - \frac{z}{ppq} = \left(\frac{dS}{dp}\right) \text{ et } y - \frac{z}{pqq} = \left(\frac{dS}{dq}\right).$$

Nunc igitur solutio ita se habebit. Introductis his ternis variabilibus p, q et z sumtaque binarum p et q functione quacunque S, capiatur:

$$x = \frac{z}{ppq} + \left(\frac{dS}{dp}\right) \text{ et } y = \frac{z}{pqq} + \left(\frac{dS}{dq}\right)$$

ac tum functio quaesita v ita definietur vt sit

$$v = \frac{3z}{pq} + p\left(\frac{dS}{dp}\right) + q\left(\frac{dS}{dq}\right) - S.$$

Vel si malimus v per ipsas tres variabiles x, y, z exprimere, ex binis aequationibus:

$$x = \frac{z}{ppq} + \left(\frac{dS}{dp}\right) \text{ et } y = \frac{z}{pqq} + \left(\frac{dS}{dq}\right)$$

quae-

quaerantur valores ipsarum p et q, quibus in functione S substitutis erit

$$v = px + qy + \frac{z}{pq} - S$$

sicque quaesito erit satisfactum.

Coroll. 1.

486. Si functio S sumatur quantitas constans C, ob

$$ppq = \frac{z}{x} \text{ et } pqq = \frac{z}{y} \text{ erit}$$

$$pq = \sqrt[3]{\frac{zz}{xy}}, \text{ hincque}$$

$$p = \sqrt[3]{\frac{yz}{xx}} \text{ et } q = \sqrt[3]{\frac{xz}{yy}}; \text{ vnde fit}$$

$$v = 3\sqrt[3]{xyz} - C$$

qui est valor particularis problemati satisfaciens.

Coroll. 2.

487. Quoniam in conditione praescripta

$$pqr = 1 \text{ seu } \left(\frac{dv}{dx}\right)\left(\frac{dv}{dy}\right)\left(\frac{dv}{dz}\right) = 1,$$

tantum differentialia trium variabilium x, y et z occurrunt, eas quantitatibus constantibus quibusuis augere licet, vnde nascitur solutio aliquanto latius patens

$$v = 3\sqrt[3]{(x+a)(y+b)(z+c)} - C.$$

Scholion 1.

488. Alius datur praeterea casus facilem euolutionem admittens ponendo $S = 2c\sqrt{pq}$, vnde colligitur

$$p = \frac{\sqrt{y}}{\sqrt{x}}\sqrt[3]{\frac{z}{\sqrt{xy}-c}} \text{ et } q = \frac{\sqrt{x}}{\sqrt{y}}\sqrt[3]{\frac{z}{\sqrt{xy}-c}},$$

ideoque $S = 2c\sqrt[3]{\frac{z}{\sqrt{xy}-c}}$.

Assequimur ergo

$$v = 3\sqrt[3]{z(\sqrt{xy}-c)^2};$$

et permutandis variabilibus simili modo habebimus:

$$v = 3\sqrt[3]{y(\sqrt{xz}-b)^2} \text{ et } v = 3\sqrt[3]{x(\sqrt{yz}-a)^2}$$

vbi porro pro x, y, z scribere licet $x+f$, $y+g$, $z+h$. Ceterum patet solutionem generalem perinde succedere; si quantitas r functioni cuicunque ipsarum p et q aequari debeat, seu si inter p, q, r aequatio quaecunque proponatur.

Scholion 2.

489. Quodsi enim posito

$$dv = p\,dx + q\,dy + r\,dz$$

inter binas formulas

$$p = \left(\frac{dv}{dx}\right),\ q = \left(\frac{dv}{dy}\right),\ r = \left(\frac{dv}{dz}\right)$$

aequatio proponatur quaecunque, quae differentiata praebeat:

$$P\,dp + Q\,dq + R\,dr = 0;$$

tum

tum facto

$$S = \int (x dp + y dq + z dr)$$

vt fit

$$v = px + qy + rz - S,$$

sumatur functio quaecunque trium quantitatum p, q, r, quae sit V haecque differentiata praebeat

$$dV = L dp + M dq + N dr$$

tum vero est

$$0 = P u dp + Q u dq + R u dr$$

ideoque

$$dV = (L + Pu) dp + (M + Qu) dq + (N + Ru) dr$$

quae forma ob nouam introductam variabilem u latissime patet. Statuatur iam $S = V$, fietque

$$x = L + Pu; \; y = M + Qu; \; z = N + Ru$$

ita vt nunc praeter variabiles p, q, r, quarum vna per binas reliquas datur, noua habeatur u, ex quibus iam tres x, y et z ita definiuimus, vt per eas vicissim hae p, q, r et u determinentur, tum vero erit

$$v = px + qy + rz - V.$$

Quare pro V sumta quacunque functione trium quantitatum p, q, r, inter quas eiusmodi conditio praescribitur, vt sit

$$P dp + Q dq + R dr = 0,$$

sumatur:

$$x = Pu + \left(\frac{dV}{dp}\right); \; y = Qu + \left(\frac{dV}{dq}\right); \; z = Ru + \left(\frac{dV}{dr}\right)$$

erit-

eritque:

$$v = (Pp+Qq+Rr)u + p\left(\frac{dV}{dp}\right) + q\left(\frac{dV}{dq}\right) + r\left(\frac{dV}{dr}\right) - V$$

quae solutio praecedenti ideo est anteferenda, quod in hanc tres quantitates p, q, r aequaliter ingrediuntur.

Problema 84.

490. Si posito

$$dv = p\,dx + q\,dy + r\,dz$$

haec conditio praescribatur vt esse debeat $pqr = \frac{v^3}{xyz}$, naturam functionis v definire.

Solutio.

Ponamus $p = \frac{Pv}{x}$, $q = \frac{Qv}{y}$, $r = \frac{Rv}{z}$, et ob conditionem praescriptam debet esse $PQR = 1$; tum vero erit

$$\frac{dv}{v} = \frac{P\,dx}{x} + \frac{Q\,dy}{y} + \frac{R\,dz}{z}.$$

Statuamus nunc

$$lv = V;\ lx = X;\ ly = Y;\ lz = Z$$

et habebimus hanc aequationem

$$dV = P\,dX + Q\,dy + R\,dZ$$

pro qua esse debet $PQR = 1$, quae quaestio cum non discrepet a problemate praecedente, eadem solutio huc quoque facillime transferetur.

Scho-

Scholion.

491. Plures casus, quos forte in hoc capite expedire liceat, hic non euoluo, cum quia vsus nondum perspicitur, tum vero imprimis, quoniam huius partis calculi integralis prorsus adhuc incognitae prima tantum principia adumbrare constitui. Pro formulis autem differentialibus altiorum graduum, quae in conditionem praescriptam ingrediantur, vix quicquam proferre licet, praeter quasdam obseruationes ad aequationes homogeneas pertinentes, quibus ergo hanc partem calculi integralis sum finiturus, simulque toti operi finem impositurus.

CAPVT IV.

DE AEQVATIONVM DIFFERENTIALIVM HOMOGENEARVM RESOLVTIONE.

Problema 85.

492.

Si v aequetur functioni cuicunque binarum quantitatum t et u, ita per tres variabiles x, y et z determinatarum vt fit

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z$$

eius formulas differentiales omnium graduum inde definire.

Solutio.

Cum v fit functio quantitatum

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z,$$

eius formulae differentiales ex his duabus variabilibus natae innotescent, scilicet:

$$\left(\frac{dv}{dt}\right);\ \left(\frac{dv}{du}\right);\ \left(\frac{ddv}{dt^2}\right);\ \left(\frac{ddv}{dt\,du}\right);\ \left(\frac{ddv}{du^2}\right) \text{ etc.}$$

hinc autem statim colligimus:

$$\left(\frac{dv}{dx}\right) = \alpha\left(\frac{dv}{dt}\right);\ \left(\frac{dv}{dy}\right) = \gamma\left(\frac{dv}{du}\right);\ \left(\frac{dv}{dz}\right) = \beta\left(\frac{dv}{dt}\right) + \delta\left(\frac{dv}{du}\right)$$

formu-

formulas ſcilicet differentiales primi gradus. Pro formulis autem differentialibus ſecundi gradus adipiſcimur :

$$\left(\frac{ddv}{dx^2}\right)=\alpha\alpha\left(\frac{ddv}{dt^2}\right);\ \left(\frac{ddv}{dy^2}\right)=\gamma\gamma\left(\frac{ddv}{du^2}\right)$$

$$\left(\frac{ddv}{dz^2}\right)=\beta\beta\left(\frac{ddv}{dt^2}\right)+2\beta\delta\left(\frac{ddv}{dt\,du}\right)+\delta\delta\left(\frac{ddv}{du^2}\right)$$

$$\left(\frac{ddv}{dx\,dy}\right)=\alpha\gamma\left(\frac{ddv}{dt\,du}\right);\ \left(\frac{ddv}{dx\,dz}\right)=\alpha\beta\left(\frac{ddv}{dt^2}\right)+\alpha\delta\left(\frac{ddv}{dt\,du}\right)$$

et $\left(\frac{ddv}{dy\,dz}\right)=\beta\gamma\left(\frac{ddv}{dt\,du}\right)+\gamma\delta\left(\frac{ddv}{du^2}\right)$.

Simili modo ad tertium gradum aſcendimus :

$$\left(\frac{d^3v}{dx^3}\right)=\alpha^3\left(\frac{d^3v}{dt^3}\right);\ \left(\frac{d^3v}{dy^3}\right)=\gamma^3\left(\frac{d^3v}{du^3}\right)$$

$$\left(\frac{d^3v}{dz^3}\right)=\beta^3\left(\frac{d^3v}{dt^3}\right)+3\beta^2\delta\left(\frac{d^3v}{dt^2\,du}\right)+3\beta\delta^2\left(\frac{d^3v}{dt\,du^2}\right)+\delta^3\left(\frac{d^3v}{du^3}\right)$$

$$\left(\frac{d^3v}{dx^2\,dy}\right)=\alpha\alpha\gamma\left(\frac{d^3v}{dt^2\,du}\right);\ \left(\frac{d^3v}{dx\,dy^2}\right)=\alpha\gamma\gamma\left(\frac{d^3v}{dt\,du^2}\right)$$

$$\left(\frac{d^3v}{dx^2\,dz}\right)=\alpha\alpha\beta\left(\frac{d^3v}{dt^3}\right)+\alpha\alpha\delta\left(\frac{d^3v}{dt^2\,du}\right)$$

$$\left(\frac{d^3v}{dy^2\,dz}\right)=\beta\gamma\gamma\left(\frac{d^3v}{dt\,du^2}\right)+\gamma\gamma\delta\left(\frac{d^3v}{du^3}\right)$$

$$\left(\frac{d^3v}{dx\,dz^2}\right)=\alpha\beta\beta\left(\frac{d^3v}{dt^3}\right)+2\alpha\beta\delta\left(\frac{d^3v}{dt^2\,du}\right)+\alpha\delta\delta\left(\frac{d^3v}{dt\,du^2}\right)$$

$$\left(\frac{d^3v}{dy\,dz^2}\right)=\beta\beta\gamma\left(\frac{d^3v}{dt^2\,du}\right)+2\beta\gamma\delta\left(\frac{d^3v}{dt\,du^2}\right)+\gamma\delta\delta\left(\frac{d^3v}{du^3}\right)$$

$$\left(\frac{d^3v}{dx\,dy\,dz}\right)=\alpha\beta\gamma\left(\frac{d^3v}{dt^2\,du}\right)+\alpha\gamma\delta\left(\frac{d^3v}{dt\,du^2}\right)$$

vnde facile patet quomodo has formulas differentiales ad altiores gradus continuari oporteat.

Scholion 1.

493. Hoc problema fortaſſe generalius concipi debuiſſe videbitur, quantitates t et u ita per

 tres

tres variabiles x, y, z definiendo, vt esset

$$t = \alpha x + \beta y + \gamma z \text{ et } u = \delta x + \varepsilon y + \zeta z$$

verum cum haec hypothesis in eum tantum finem sit facta, vt v fieret functio ipsarum t et u, euidens tum quoque v spectari posse vt functionem harum duarum quantitatum $\varepsilon t - \beta u$ et $\delta t - \alpha u$, quarum illa ab y haec vero ab x erit libera. Quocirca hypothesis assumta latissime patere est censenda, exceptio tamen forte hinc admittenda videbitur, si fuerit $t = x + z$ et $u = x - z$ quia hic ipsius u valor non continetur, verum etiam hoc casu quantitas v vt functio ipsarum $t + u$ et $t - u$ spectata fiet functio ipsarum x et z qui casus vtique in hypothesi continetur, sumtis $\beta = 0$ et $\gamma = 0$.

Scholion 2.

494. Hoc problema ideo praemisi, quia alias aequationes differentiales tractare hic non sustineo, nisi quibus eiusmodi valor satisfacit, vt v aequetur functioni cuicunque binarum nouarum variabilium t et u, quae ab principalibus x, y, z ita pendeant, vt sit quemadmodum assumsi

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z.$$

Huiusmodi autem aequationes, quibus hoc modo satisfieri potest, esse homogeneas, facile patet, ita vt aequatio resoluenda constet nonnisi formulis differentialibus eiusdem gradus, singulis per constantes quantitates multiplicatis, et inter se additis, qua

qua appellatione aequationum homogenearum iam in parte praecedente ſum vſus. Propoſita ergo huiusmodi aequatione homogenea, loco ſingularum formularum differentialium per elementa dx, dy, dz formatarum ſubſtituantur valores hic inuenti per elementa dt et du formati, et tum ſingula membra, quatenus certam formulam differentialem ex elementis dt et du natam complectuntur, ſeorſim ad nihilum redigantur; indeque rationes $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ determinentur; quandoquidem quaeſtio non tam circa has ipſas quantitates, quam earum rationes verſatur. Quoniam igitur duae tantum res inueſtigationi relinquuntur, ſi pluribus aequationibus fuerit ſatisfaciendum, eiusmodi aequationes homogeneae hac ratione reſolui neqneunt, niſi caſu plures illae aeaequationes ad duas tantum reuocentur, id quod in ſequentibus clarius explicabitur.

Problema 86.

495. Propoſita aequatione homogenea primi gradus:

$$A\left(\frac{dv}{dx}\right)+B\left(\frac{dv}{dy}\right)+C\left(\frac{dv}{dz}\right)=0$$

inueſtigare naturam functionis v trium variabilium x, y et z.

Solutio.

Fingatur $v=\Gamma:(t \text{ et } u)$ exiſtente

$$t=\alpha x+\beta z \text{ et } u=\gamma y+\delta z$$

 et

et facta substitutione ex probl. praeced. aequatio nostra in duas partes diuidetur:

$$\left(\frac{dv}{dt}\right)(A\alpha+C\beta)+\left(\frac{dv}{du}\right)(B\gamma+C\delta)=0$$

quarum vtraque seorsim ad nihilum reducta praebet

$$\frac{\beta}{\alpha}=\frac{-A}{C} \text{ et } \frac{\delta}{\gamma}=\frac{-B}{C}$$

vnde fit

$$t=Cx-Az \text{ et } u=Cy-Bz.$$

Quare aequationis propositae integrale completum erit

$$v=\Gamma:(\overline{Cx-Az} \text{ et } \overline{Cy-Bz})$$

quod etiam concinnius ita exhiberi potest:

$$v=\Gamma:\left(\frac{x}{A}-\frac{z}{C} \text{ et } \frac{y}{B}-\frac{z}{C}\right).$$

Coroll. 1.

496. Permutandis variabilibus hoc integrale etiam ita exprimi posse euidens est

$$v=\Gamma:\left(\frac{x}{A}-\frac{y}{B} \text{ et } \frac{y}{B}-\frac{z}{C}\right) \text{ vel}$$

$$v=\Gamma:\left(\frac{x}{A}-\frac{y}{B} \text{ et } \frac{x}{A}-\frac{z}{C}\right)$$

quoniam est

$$\frac{x}{A}-\frac{y}{B}=\left(\frac{x}{A}-\frac{z}{C}\right)-\left(\frac{y}{B}-\frac{z}{C}\right)$$

Coroll. 2.

Coroll. 2.

497. Quin etiam constitutis ex aequatione proposita his tribus formulis:

$$\frac{x}{A}-\frac{y}{B};\ \frac{x}{A}-\frac{z}{C};\ \frac{y}{B}-\frac{z}{C}$$

functio quaecunque ex iis vtcunque conflata valorem idoneum pro v suppeditabit. Quoniam enim harum binarum formularum vnaquaeque est differentia binarum reliquarum, talis functio duas tantum variabiles complecti est censenda.

Coroll. 3.

498. Perinde est quanam harum trium formarum integralium vtamur, quando autem binae nouae variabiles t et u inter se fuerint aequales, tum alia est vtendum. Veluti si esset $C=0$ prima forma $v=\Gamma:(z \text{ et } z)$, vt potefunctio solius z foret inutilis, et integrale completum esset futurum:

$$v=\Gamma:\left(\frac{x}{A}-\frac{y}{B} \text{ et } z\right) \text{ seu}$$

$$v=\Gamma:\left(\overline{Bx-Ay} \text{ et } z\right).$$

Problema 87.

499. Proposita aequatione homogenea secundi gradus:

$$A\left(\frac{ddv}{dx^2}\right)+B\left(\frac{ddv}{dy^2}\right)+C\left(\frac{ddv}{dz^2}\right)+2D\left(\frac{ddv}{dxdy}\right)+2E\left(\frac{ddv}{dxdz}\right)$$
$$+2F\left(\frac{ddv}{dydz}\right)=0$$

casus

casus inuestigare, quibus eius integrale hac forma $\Gamma:(t \text{ et } u)$ exprimi potest existente

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z.$$

Solutio.

Facta substitutione secundum formulas in probl. 85. traditas aequatio proposita in tria membra sequentia resoluetur:

$$\left.\begin{array}{l} (\frac{ddv}{dt^2})(A\alpha\alpha + C\beta\beta + 2E\alpha\beta) \\ (\frac{ddv}{dtdu})(2C\beta\delta + 2D\alpha\gamma + 2E\alpha\delta + 2F\beta\gamma) \\ (\frac{ddv}{du^2})(B\gamma\gamma + C\delta\delta + 2F\gamma\delta) \end{array}\right\} = 0$$

quorum singula seorsim nihilo debent aequari. At primum praebet

$$\frac{\beta}{\alpha} = \frac{-E + \surd(EE - AC)}{C},$$

vltimum vero

$$\frac{\delta}{\gamma} = \frac{-F + \surd(FF - BC)}{C}$$

qui valores in media, quae ita referatur

$$\frac{C\beta\delta}{\alpha\gamma} + D + \frac{E\delta}{\gamma} + \frac{F\beta}{\alpha} = 0$$

substituti suppeditant hanc aequationem:

$$EF - CD = \surd(EE - AC)(FF - BC)$$

qua aequatione conditio inter coefficientes A, B, C, D, E, F continetur, vt solutio hic applicata locum inuenire possit. Haec autem aequatio euoluta dat

$$CCDD - 2CDEF + BCEE + ACFF - ABCC = 0$$

vnde

vnde fit

$$C=\frac{2DEF-BEE-AFF}{DD-AB},$$

quia factor C per multiplicationem eſt ingreſſus. Quoties autem haec conditio habet locum vt ſit:

$$AFF+BEE+CDD=ABC+2DEF$$

toties haec expreſſio algebraica ex aequatione propoſita formanda

$$Axx+Byy+Czz+2Dxy+2Exz+2Fyz$$

in duos factores poteſt reſolui, neque ergo aliis caſibus ſolutio hic adhibita locum habere poteſt. Quo ergo hos caſus ſolutionem admittentes rite euoluamus, ponamus huius formae factores eſſe:

$$(ax+by+cz)(fx+gy+hz)$$

quod ergo eueniet, ſi fuerit

$$A=af;\quad B=bg;\quad C=ch$$
$$2D=ag+bf;\ 2E=ah+cf;\ 2F=bh+cg$$

vnde vtique fit

$$AFF+BEE+CDD=ABC+2DEF.$$

Hinc autem pro ſolutione colligitur:

vel $\frac{\beta}{\alpha}=\frac{-a}{c}$ vel $\frac{\beta}{\alpha}=\frac{-f}{h}$ et

vel $\frac{\delta}{\gamma}=\frac{-b}{c}$ vel $\frac{\delta}{\gamma}=\frac{-g}{h}$

vbi obſeruari oportet pro fractionibus $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ valores ſibi ſubſcriptos coniungi oportere, ita vt ſit

vel $t=cx-az$ et $u=cy-bz$

vel $t=hx-fz$ et $u=hy-gz$.

Quo-

Quocirca pro his casibus solutionem admittentibus integrale completum erit

$$v = \Gamma:(\overline{cx-az} \text{ et } \overline{cy-bz}) + \Delta:(\overline{bx-fz} \text{ et } \overline{by-gz})$$

seu

$$v = \Gamma:(\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}) + \Delta:(\tfrac{x}{f} - \tfrac{z}{b} \text{ et } \tfrac{y}{g} - \tfrac{z}{b}).$$

Coroll. 1.

500. Hoc ergo modo aliae aequationes homogeneae secundi gradus resolui nequeunt, nisi quae in hac forma continentur:

$$af(\tfrac{ddv}{dx^2}) + bg(\tfrac{ddv}{dy^2}) + cb(\tfrac{ddv}{dz^2}) + (ag+bf)(\tfrac{ddv}{dxdy}) + (ab+cf)(\tfrac{ddv}{dxdz}) + (bb+cg)(\tfrac{ddv}{dydz}) = 0$$

tum vero integrale completum erit

$$v = \Gamma:(\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}) + \Delta:(\tfrac{x}{f} - \tfrac{z}{b} \text{ et } \tfrac{y}{g} - \tfrac{z}{b}).$$

Coroll. 2.

501. Quo autem facilius dignoscatur, vtrum aequatio quaepiam proposita:

$$A(\tfrac{ddv}{dx^2}) + B(\tfrac{ddv}{dy^2}) + C(\tfrac{ddv}{dz^2}) + 2D(\tfrac{ddv}{dxdy}) + 2E(\tfrac{ddv}{dxdz}) + 2F(\tfrac{ddv}{dydz}) = 0$$

eo reduci possit nec ne? formetur inde haec forma algebraica

$$Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz$$

quae si resolui patiatur in duos factores rationales

$$(ax + by + cz)(fx + gy + bz)$$

eius

eius integrale completum hinc ſtatim exhiberi poteſt.

Coroll. 3.

502. Vnicus tantum caſus quo duo iſti factores inter ſe fiunt aequales, exceptionem poſtulat, quoniam tum binae functiones inuentae in vnam coaleſcerent. Verum ex ſuperioribus colligitur, ſi hoc eueniat vt ſit $f=a$, $b=g$ et $b=c$ integrale completum ita exprimi

$$z=x\Gamma:\left(\frac{x}{a}-\frac{z}{c}\text{ et }\frac{y}{b}-\frac{z}{c}\right)+\Delta:\left(\frac{x}{a}-\frac{z}{c}\text{ et }\frac{y}{b}-\frac{z}{c}\right).$$

Scholion 1.

503. Quibus ergo caſibus aequatio homogenea ſecundi gradus reſolutionem admittit, iis quoque in ſe complectitur duas aequationes homogeneas primi gradus

$$a\left(\frac{dv}{dx}\right)+b\left(\frac{dv}{dy}\right)+c\left(\frac{dv}{dz}\right)=0 \text{ et}$$
$$f\left(\frac{dv}{dx}\right)+g\left(\frac{dv}{dy}\right)+b\left(\frac{dv}{dz}\right)=0$$

quippe quarum vtraque illi ſatisfacit, et harum integralia completa iunctim ſumta illius integrale completum ſuppeditant. Hinc alia via aperitur aequationum homogenearum ſecundi gradus integralia inueniendi fingendo aequationem primi gradus ipſis ſatisfacientem:

$$a\left(\frac{dv}{dx}\right)+b\left(\frac{dv}{dy}\right)+c\left(\frac{dv}{dz}\right)=0$$

 tum

tum ex hac per triplicem differentiationem tres nouae formentur:

$$a\left(\frac{ddv}{dx^2}\right)+b\left(\frac{ddv}{dxdy}\right)+c\left(\frac{ddv}{dxdz}\right)=0$$
$$a\left(\frac{ddv}{dxdy}\right)+b\left(\frac{ddv}{dy^2}\right)+c\left(\frac{ddv}{dydz}\right)=0$$
$$a\left(\frac{ddv}{dxdz}\right)+b\left(\frac{ddv}{dydz}\right)+c\left(\frac{ddv}{dz^2}\right)=0$$

quarum prima per f, secunda per g et tertia per h multiplicatae et in vnam summam collectae, ipsam illam aequationem generalem producunt, cuius integrale supra exhibuimus. Ea ergo quasi productum ex binis aequationibus homogeneis primi gradus spectari poterit, ex quibus coniunctis integrale completum formatur.

Scholion 2.

504. Infinitae ergo aequationes homogeneae secundi gradus hic excluduntur, quae hoc modo integrationem respuunt, seu ad aequationes primi gradus reduci nequeunt; qui casus exclusi omnes ex hoc criterio agnoscuntur, si non fuerit

$$AFF+BEE+CDD=ABC+2DEF.$$

Huius generis est ista aequatio $\left(\frac{ddv}{dxdy}\right)=\left(\frac{ddv}{dz^2}\right)$, quae ergo tale integrale, cuiusmodi hic assumsimus non admittit, neque etiam alia patet via eius integrale completum inuestigandi. Integralia autem particularia facile innumera exhiberi possunt, et quae adeo functiones arbitrarias complectuntur, sed tantum vnius

vnius quantitatis variabilis, quae in praesenti instituto nonnisi integralia particularia constituere sunt censendae. Si enim ponatur

$$v = \Gamma : (\alpha x + \beta y + \gamma z)$$

facta substitutione fieri debet $\alpha\beta = \gamma\gamma$, seu sumto $\gamma = 1$, debet esse $\alpha\beta = 1$; quare innumerabiles adeo huiusmodi formulae coniunctae satisfaciunt vt sit

$$v = \Gamma : \left(\frac{\alpha}{\beta} x + \frac{\beta}{\alpha} y + z\right) + \Delta : \left(\frac{\gamma}{\delta} x + \frac{\delta}{\gamma} y + z\right)$$
$$\Sigma : \left(\frac{\varepsilon}{\zeta} x + \frac{\zeta}{\varepsilon} y + z\right) \text{ etc.}$$

vbi pro α, β, γ, δ etc. numeros quoscunque accipere licet, quamuis autem infinitae huiusmodi formulae diuersae coniungantur, tamen integrale nonnisi pro particulari haberi potest. Ex quo intelligitur integrationem completam istius aequationis $\left(\frac{ddv}{dxdy}\right) = \left(\frac{ddv}{dz^2}\right)$ maximi esse momenti, methodumque eo perueniendi fines analyseos non mediocriter esse prolaturam. Aequationes autem homogeneae tertii gradus multo maiorem restrictionem exigunt, vt integratio completa hoc modo succedat, vti sequenti problemate ostendetur.

Problema 83.

505. Aequationum homogenearum tertii gradus eos casus definire, quibus integrale completum per formam assumtam exhiberi, seu ad formam aequationum homogenearum primi gradus reduci potest.

Solutio.

In aequatione homogenea tertii gradus fingatur contineri haec primi gradus

$$a\left(\frac{dv}{dx}\right)+b\left(\frac{dv}{dy}\right)+c\left(\frac{dv}{dz}\right)=0,$$

quae vt satisfaciat aequationi tertii gradus:

$$\left.\begin{aligned}A\left(\frac{d^3v}{dx^3}\right)+B\left(\frac{d^3v}{dy^3}\right)+C\left(\frac{d^3v}{dz^3}\right)+D\left(\frac{d^3v}{dx^2dy}\right)&+E\left(\frac{d^3v}{dxdy^2}\right)\\ +F\left(\frac{d^3v}{dx^2dz}\right)&+G\left(\frac{d^3v}{dx\,dz^2}\right)\\ +H\left(\frac{d^3v}{dy^2dz}\right)&+I\left(\frac{d^3v}{dydz^2}\right)\\ &+K\left(\frac{d^3v}{dxdydz}\right)\end{aligned}\right\}=0$$

necesse est vt expressio haec algebraica:

$$\begin{aligned}Ax^3+By^3+Cz^3+Dxxy+Fxxz+Hyyz+Kxyz&\\ +Exyy+Gxzz+Iyzz&\end{aligned}$$

factorem habeat $ax+by+cz$; nisi autem alter factor denuo in duos simplices sit resolubilis, ad aequationem homogeneam secundi gradus referetur, quae solutionem respuit. Quare vt integratio completa succedat necesse est, istam expressionem tribus constare factoribus simplicibus, qui sint

$$(ax+by+cz)(fx+gy+hz)(kx+my+nz)$$

hincque aequationis generalis coefficientes ita se habebunt:

$$\begin{array}{lll} A=afk; & D=afm+agk+bfk; & H=bgn+bhm+cgm \\ B=bgm & E=agm+bfm+bgk; & I=bhn+cgn+chm \\ C=chn; & F=afn+ahk+cfk; & K=agn+ahm+bfn \\ & G=ahn+cfn+chk; & \quad +bhk+cfm+cgk \end{array}$$

ac

ac tum integrale completum erit

$$v = \Gamma:\left(\frac{x}{a}-\frac{z}{c}\text{ et }\frac{y}{b}-\frac{z}{c}\right)+\Delta:\left(\frac{x}{f}-\frac{z}{h}\text{ et }\frac{y}{g}-\frac{z}{h}\right)$$
$$+\Sigma:\left(\frac{x}{k}-\frac{z}{n}\text{ et }\frac{y}{m}-\frac{z}{n}\right)$$

quilibet ſcilicet factor ſimplex praebet functionem arbitrariam duarum variabilium.

Coroll. 1.

506. In qualibet harum functionum variabiles x, y, z inter ſe permutare licet; quin etiam quaelibet quaſi ex tribus variabilibus conflata ſpectari poteſt, prima nempe ex his

$$\frac{x}{a}-\frac{y}{b};\ \frac{y}{b}-\frac{z}{c};\text{ et }\frac{z}{c}-\frac{x}{a},$$

ſimilique modo de ceteris.

Coroll. 2.

507. Si duo factores fuerint aequales $f=a$, $g=b$, $h=c$ quo caſu duae priores functiones in vnam coaleſcerent, earum loco ſcribi debet

$$x\Gamma:\left(\frac{x}{a}-\frac{z}{c}\text{ et }\frac{y}{b}-\frac{z}{c}\right)+\Delta:\left(\frac{x}{a}-\frac{z}{c}\text{ et }\frac{y}{b}-\frac{z}{c}\right),$$

at ſi omnes tres fuerint aequales, vt inſuper ſit $k=a$, $m=b$, $n=c$ integrale completum erit:

$$v=xx\Gamma:\left(\frac{x}{a}-\frac{z}{c}\text{ et }\frac{y}{b}-\frac{z}{c}\right)+x\Delta:\left(\frac{x}{a}-\frac{z}{c}\text{ et }\frac{y}{b}-\frac{z}{c}\right)$$
$$+\Sigma:\left(\frac{x}{a}-\frac{z}{c}\text{ et }\frac{y}{b}-\frac{z}{c}\right).$$

Coroll. 3.

508. Quemadmodum hic duas priores partes per xx et x multiplicauimus, ita eas quoque per yy et y item zz et z multiplicare poſſemus, perinde enim eſt quanam variabili hic vtamur, dum ne ſit ea quae forte ſola poſt ſignum functionis occurrit, ſcilicet

licet si esset $a=0$, et functiones quantitatum x et $\frac{y}{b}-\frac{z}{c}$ capi debeant, tum multiplicatores xx et x excludi deberent.

Scholion 1.

509. Simili modo patet aequationes homogeneas quarti gradus hac methodo resolui non posse, nisi in quatuor eiusmodi aequationes simplices resolui, et quasi earum producta spectari queant. Etsi enim hic reuera nulla resolutio in factores locum habeat tamen ex allatis exemplis clare perspicitur, quemadmodum ex aequatione differentiali homogenea cuiuscunque gradus expressio algebraica eiusdem gradus ternas variabiles x, y, z inuoluens debeat formari; quae si in factores simplices formae $ax+by+cz$ resolui queat, simul inde aequationis differentialis integrale completum facile exhibebitur, cum quilibet factor functionem duarum variabilium suppeditet, integralis partem constituentem; ita vt etiam haec pars seorsim sumta aequationi differentiali satisfaciat et pro integrali particulari haberi possit. At si illa expressio algebraica ita fuerit comparata, vt factores quidem habeat simplices sed non tot, quot dimensiones, singuli quidem integralia particularia praebebunt, quae autem iunctim sumta non integrale completum suppeditabunt. Veluti si proponatur haec aequatio differentialis tertii gradus:

$$a\left(\frac{d^3 v}{dx^2\, dy}\right)+b\left(\frac{d^3 v}{dx\, dy^2}\right)-a\left(\frac{d^3 v}{dx\, dz^2}\right)-b\left(\frac{d^3 v}{dy\, dz^2}\right)=0$$

quia forma algebraica $axxy+bxyy-axzz-byzz$ factorem habet simplicem $ax+by$, illi vtique satisfaciet

faciet valor $v=\Gamma:(\frac{x}{a}-\frac{y}{b}\text{ et }z)$ pro integrali autem completo adhuc desunt duae functiones arbitrariae, integrale completum huius aequationis $(\frac{ddv}{dxdy})-(\frac{ddv}{dz^2})=0$ continentes, ex qua quippe alter factor $xy-zz$ illius expressionis nascitur. Quoties ergo hae expressiones algebraicae ex aequationibus differentialibus homogeneis altiorum graduum formatae resolutionem in factores, etsi non simplices, admittant; hinc saltem discimus, quomodo earum integratio ad aequationes inferiorum graduum reuocari possit, quod in huiusmodi arduis inuestigationibus sine dubio maximi est momenti.

Scholion 2.

510. Haec sunt quae de functionibus trium variabilium ex data quadam differentialium relatione inuestigandis proferre potui, in quibus vtique nonnisi prima elementa huius scientiae continentur, quorum vlterior euolutio sagacitati Geometrarum summo studio est commendanda. Tantum enim abest, vt hae speculationes pro sterilibus sint habendae, vt potius pleraque, quae adhuc in Theoria motus fluidorum desiderantur, ad has Analyseos partes sublimiores sint referenda, quarum propterea vtilitas neutiquam parti priori calculi integralis postponenda videtur. Eo magis autem hae partes posteriores excoli merentur, quod Theoria fluidorum adeo circa functiones quatuor variabilium versetur, quarum naturam ex aequationibus differentialibus secundi gradus inuestigari oportet, quam partem ob penuriam materiae ne attingere quidem volui. In hac autem Theoria resolutio huius

aequationis:

$$\left(\frac{ddv}{dt^2}\right)=\left(\frac{ddv}{dx^2}\right)+\left(\frac{ddv}{dy^2}\right)+\left(\frac{ddv}{dz^2}\right)$$

maximi eſt momenti, vbi litterae x, y, z ternas coordinatas, t vero tempus elapſum exprimunt, harumque quatuor variabilium functio quaeritur, quae loco v ſubſtituta illi aequationi ſatisfaciat. Ex hactenus autem allatis facile colligitur, integrale completum huius aequationis duas complecti debere functiones arbitrarias, quarum vtraque ſit functio trium variabilium, aliasque ſolutiones omnes minus late patentes pro incompletis eſſe habendas. Facili autem negotio innumeras ſolutiones particulares exhibere licet, veluti ſi ponamus $v=\Gamma:(\alpha x+\beta y+\gamma z+\delta t)$ reperitur: $\delta\delta=\alpha\alpha+\beta\beta+\gamma\gamma$, quod cum infinitis modis fieri poſſit infinitae huiusmodi functiones additae valorem idoneum pro v exhibebunt. Deinde etiam ſatisfaciunt iſti valores

$$v=\frac{\Gamma:(t\pm\sqrt{(xx+yy+zz)})}{\sqrt{(xx+yy+zz)}}$$

$$v=\frac{\Gamma:(x\pm\sqrt{(tt-yy-zz)})}{\sqrt{(tt-yy-zz)}}$$

$$v=\frac{\Gamma:(y\pm\sqrt{(tt-xx-zz)})}{\sqrt{(tt-xx-zz)}}$$

$$v=\frac{\Gamma:(z\pm\sqrt{(tt-xx-yy)})}{\sqrt{(tt-xx-yy)}}$$

quorum quidem inueſtigatio iam eſt difficilior. Cum autem hae functiones tantum ſint vnius variabilis integralia maxime particularia exhibent; quae adeo etiamnum forent particularia, ſi pro v functiones binarum variabilium haberentur, quales autem ne ſuſpicari quidem licet. Quare cum integrale completum duas adeo functiones arbitrarias trium variabilium complecti debeat, facile intelligitur, quantopere adhuc ab hoc ſcopo ſimus remoti.

APPEN-

APPENDIX
DE
CALCVLO
VARIATIONVM.

CAPVT I.

DE CALCVLO VARIATIONVM IN GENERE.

Definitio 1.

1.

Relatio inter binas variabiles variari dicitur, si valor, quo altera inde per alteram determinatur; incremento infinite paruo augeri concipiatur, quod incrementum variationem eius quantitatis, cui adiicitur, vocabimus.

Explicatio.

2. Primum ergo hic consideratur relatio inter binas variabiles, x et y quaecunque, aequatione quacunque inter easdem expressa, qua pro singulis

 valo-

valoribus ipſi x tributis valores ipſius y conuenientes determinentur, tum vero ſinguli valores ipſius y particulis infinite paruis vtcunque augeri concipiantur, ita vt hi valores variati a veris, quos ex relatione propoſita ſortiuntur infinite parum diſcrepent, atque hoc modo relatio illa inter x et y variari dicitur, ſimulque particulae illae infinite paruae valoribus veris ipſius y adiunctae appellantur. Imprimis autem hic notandum eſt has variationes, quibus ſinguli valores ipſius y augeri concipiuntur, neque inter ſe ſtatui aequales, neque vllo modo a ſe inuicem pendentes, ſed ita arbitrio noſtro permitti, vt omnes praeter vnam vel aliquas certis valoribus ipſius y reſpondentes plane vt nullas ſpectare liceat. Nulli ſcilicet legi hae variationes adſtrictae ſunt concipiendae, neque relatio inter x et y data vllam determinationem in iſtas variationes inferre eſt cenſenda; quas vt prorſus arbitrarias ſpectare oportet.

Coroll. 1.

3. Hinc patet variationes toto coelo differre a differentialibus etiamſi vtraque ſint infinite parua ideoque plane euaneſcant variatio enim afficit eundem valorem ipſius y, eidem valori ipſius x conuenientem, dum eius differentiale dy ſimul ſequentem valorem $x+dx$ reſpicit.

Coroll. 2.

4. Si enim ex relatione inter x et y propoſita ipſi x conueniat y, ipſi $x+dx$ vero valor ipſius y conueniens ponatur y', tum eſt $dy=y'-y$;

at

at variatio ipſius y neutiquam pendet a valore ſequente y', quin potius vtrique y et y' pro lubitu ſuam variationem ſeorſim tribuere licet.

Scholion.

5. Haec variationum idea quae per ſe tam nimis vaga quam ſterilis videri queat, maxime illuſtrabitur, ſi eius originem et quo pacto ad eam eſt peruentum, accuratius expoſuerimus. Perduxit autem eo potiſſimum quaeſtio de curuis inueniendis, quae certa quadam maximi minimiue proprietate ſint praeditae, vnde ne rem in genere conſiderando obſcuritas offundatur, problema contemplemur, quo linea curua quaeritur, ſuper qua graue delabens e dato puncto citiſſime ad aliud punctum datum deſcendat. Atque hic quidem ex natura maximorum et minimorum ſtatim conſtat, curuam ita debere eſſe comparatam, vt ſi eius loco alia curua quaecunque infinite parum ab illa diſcrepans ſubſtituatur tempus deſcenſus ſuper ea idem prorſus ſit futurum. Solutionem ergo ita inſtitui oportet, vt dum curua quaeſita tanquam data ſpectatur, calculus quoque ad aliam curuam infinite parum ab ea diſcrepantem accommodetur, indeque diſcrimen quod in temporis expreſſionem redundat, ſupputetur; tum enim hoc ipſum diſcrimen nihilo aequale poſitum naturam curuae, quaeſitae declarabit. Curuae autem iſtae infinite parum a quaeſita diſcrepantes commodiſſime ita conſiderantur, vt applicatae ſingulis abſciſſis reſpondentes particulis infinite paruis

paruis vel augeantur vel minuantur, hoc eſt, vt *variationes* recipere concipiantur. Vulgo quidem ſufficit huiusmodi variationem in vnica applicata conſtituiſſe, nihil autem impedit, quominus pluribus atque adeo omnibus applicatis tales variationes aſſignentur, cum ſemper ad eandem ſolutionem perduci ſit neceſſe. Hoc autem modo non ſolum vis methodi multo luculentius illuſtratur ſed etiam inde ſolutiones quaeſtionum huius generis pleniores obtinentur, vnde etiam quaeſtiones ad alias conditiones ſpectantes enucleare licet. Quam ob cauſam omnino neceſſarium videtur, vt calculus variationum in ampliſſima extenſione, cuius quidem eſt capax, pertractetur.

Definitio 2.

6. *Pro data relatione inter binas variabiles quantitates vtraque earum variari dicitur, ſi vtraque ſeorſim incremento infinite paruo augeri concipiatur; vnde patet quomodo intelligendum ſit ſi vtrique variabili ſua tribuatur variatio.*

Explicato.

7. Si propoſita ſit aequatio quaecunque inter binas variabiles x et y qua earum relatio mutua exprimitur, haec relatio per definitionem duplici modo variari poteſt, altero quo manentibus valoribus x, ſingulis y variatio tribuitur, altero vero quo manentibus valoribus y, ſinguli x variari concipiuntur. Nihil igitur prohibet, quo minus vtraque

que variabilis simul suas variationes recipere intelligatur, quas adeo ita capere licet, vt nullo plane nexu inter se cohaereant, duplex ergo hic variatio consideratur, cum in definitione prima vnica tantum sit admissa. Rem autem hic ita generaliter contemplamur, vt neutra variatio vlli legi sit adstricta neque etiam variationes ipsius y vllo modo a variationibus ipsius x pendeant.

Coroll. 1.

8. Ex casu ergo quo duplex variatio statuitur, casus prior tanquam species nascitur, si variationes alterius variabilis plane reiiciantur, vnde manifestum est casum definitionis secundae in se complecti casum primae.

Coroll. 2.

9. Hinc magis elucet, quemadmodum data relatio inter binas variabiles infinitis modis variari possit, simulque intelligitur, quoniam has variationes nulli legi adstrictas assumimus, omnes omnino illius relationis variationes possibiles hac ratione indicari.

Scholion 1.

10 Variationes quidem alterutri tantum variabili inductae iam omnes variationes possibiles, quae in propositam relationem inter binas variabiles cadere possunt, comprehendunt, vt superfluum

videri possit calculum ad duplicem variationem accommodari, verum si indolem rei, vsumque cui destinatur, attentius contemplemur, duplicis variationis consideratio neutiquam superuacanea deprehendetur, id quod per Geometriam euidentissime sequentem in modum illustrabitur. Cum relatio quaecunque inter binas variabiles distinctissime per lineam curuam in plano descriptam repraesentetur, sit AYM linea curua, aequatione inter coordinatas
Fig. 1. $AX = x$ et $XY = y$ definita, quae ergo datam illam relationem exhibeat, iam igitur quaelibet linea curua alia Aym ab illa infinite parum discrepans relationem illam variatam repraesentabit, quae quomodocunque se habeat; semper ita considerari potest, vt eidem abscissae $AX = x$ conueniat applicata variata Xv existente particula Xv eius variatione, quae consideratio quoque pro plerisque circa maxima et minima prolatis quaestionibus sufficit, vbi adeo curua AM in nonnullis tantum elementis variari solet concipi. At si quaestio ita sit comparata vt inter omnes curuas, quas a dato puncto A ad datam quampiam curuam CD vsque ducere licet, ea definiatur AYM cui maximi minimiue proprietas quaedam conueniat, tum eadem proprietas in aliam quamcunque curuam proximam Aym etiam in alio lineae CD puncto m terminatam aeque competere debet, sicque pro vltimo curuae quaesitae puncto M tam abscissa AP quam applicata PM variationem recipere est censenda, et huiusmodi quidem, quae naturae lineae

CD fit confentanea. Quo igitur calculus ad talem variationem vltimo elemento inductam accommodari queat, omnino necesse est, vt pro singulis curuae AM punctis intermediis Y generalissime tam abscissae $AX = x$ quam applicatae $XY = y$ variationes tribuantur quaecunque, illiusque variatio statuatur particula Xx huius vero $= xy - XY$, ex quo indoles simulque vsus huiusmodi duplicis variationis clarissime perspicitur.

Scholion 2.

11. Quemadmodum consideratio vltimi puncti curvae inuestigandae nobis hanc insignem dilucidationem suppeditauit ita etiam subinde primo puncto variationem tribui oportet. Veluti si inter omnes lineas, quas a data quadam curua AB ad aliam quandam Fig. 2. itidem datam CD ductas concipere licet ea sit quaerenda, quae maximi minimiue cuiuspiam proprietate sit praedita, tum multo magis erit necessarium tam singulis abscissis AX quam applicatis XY variationes quascunque nulla lege adstrictas in calculo assignari, vt deinceps tam ad initii G curuae quaesitae, quam eius finis M variationem transferri possint. Quanquam autem haec illustratio ex Geometria est desumta, tamen facile intelligitur ideam variationum inde petitam multo latius patere, atque in Analysi absoluta summo vsu non esse carituram. Celeberrimus autem *de la Grange* Acutissimus

mus Geometra Taurinensis, cui primas speculationes de calculo variationum acceptas referre debemus hanc methodum adeo ingeniosissime transtulit ad lineas non continuas veluti ad polygonorum genus referendas, in quo negotio hae duplices variationes ipsi summam praestiterunt vtilitatem.

Definitio 3.

12. *Relatio inter tres variabiles, duabus aequationibus determinata, variari dicitur, si earum vel vna, vel duae, vel omnes tres particulis infinite paruis augeantur, quae earum variationes appellantur.*

Explicatio.

13. Cum tres proponantur variabiles quantitates veluti x, y et z, inter quas duae aequationes dari concipiuntur, ex vna quaque earum binas reliquas determinare licet, ita vt tam y quam z tanquam functio ipsius x spectari possit. Hoc autem modo definiri solet linea curua non in eodem plano descripta, dum singula eius puncta per has ternas coordinatas x, y et z more solito assignantur. Quodsi iam talis curua alia quacunque sibi proxima comitetur, vt differentia sit infinite parua, haec noua curua propositae erit variata, ac relatio illa inter ternas variabiles x, y, z variata eius naturam exprimere est concipienda. Ex quo prout bina puncta proxima alterum in ipsa curua proposita, alterum in

in variata comitante assumtum inter se comparantur, fieri potest vt pro variata vel omnes tres coordinatae prodeant diuersae, vel duae tantum, vel saltem vnica harumque differentiae a coordinatis principalis curuae earum variationes repraesentabunt; quas autem hic ita generalissime contemplari conuenit, vt ad omnes omnino curuas proximas extendantur, siue eae per totum tractum a curua proposita fuerint diuersae, siue tantum in quibusdam portionibus ab ea aberrent, ita vt etiam lineae non continuae dummodo principali sint proximae; hinc non excludantur. Neque enim hae curuae variatae vlli continuitatis legi sunt adstringendae, vt omnes plane curuas possibiles infinite parum a principali aberrantes in se complectantur.

Coroll. 1.

14. Cum puncto ergo quocunque curuae propositae seu principalis comparatur punctum quodpiam curuae variatae infinite parum ab illo dissitum, et hincque coordinatarum variationes definiri intelliguntur.

Coroll. 2.

15. Quia porro ex assumta variabili vna x, binae reliquae y et z ideoque punctum curuae propositae determinatur, etiam variationes singularum coordinatarum tanquam functiones ipsius x spectare licet, dummodo earum quantitas vt infinite parua spectetur.

Coroll. 3.

16. Tres ergo quascunque functiones ipsius x vtcunque inter se diuersas concipere licet, quae per factores infinite paruos multiplicatae idoneae erunt ad ternas variationes coordinatarum repraesentandas. Quod idem de ternis quibuscunque variabilibus est tenendum etiamsi non ad geometriam referantur.

Coroll. 4.

17. Simili quoque modo si relatio tantum inter duas variabiles proponatur, earum variationes tanquam functiones alterius variabilis spectari possunt, modo sint infinitae paruae, seu quod eodem redit per quantitatem infinite paruam multiplicatae.

Scholion 1.

18. Consideratio autem geometrica maxime est idonea ad has speculationes illustrandas, quae in genere consideratae nimis abstractae atque etiam vagae videri queant. Casus igitur trium variabilium quarum relatio duabus aequationibus definiri assumitur, luculentissime per curuam non in eodem plano descriptam explicatur, dum illis variabilibus ternae coordinatae designantur. Quodsi enim de huiusmodi curuis quaestio instituatur, vt inter eas definiatur ea quae maximi minimiue proprietate quapiam sit praedita necesse est vt eadem proprietas in omnes alias curuas ab ea infinite parum aber-

rantes

rantes aeque competat, id quod ex variationibus debite in calculum introductis est diiudicandum. Cuinam autem vsui summa generalitas in variationibus hic stabilita sit futura, inde intelligere licet, si loco duarum curuarum AB et CD datae sint duae quaecunque superficies a quarum illa ad hanc eiusmodi lineam curuam duci oporteat, quae maximi minimiue quapiam gaudeat proprietate. Tum enim ternarum coordinatarum variationes ita generales considerari oportet, vt curuae quaesitae puncto ad initium in superficiem AB translato variationes ibi ad eandem superficiem accommodari possint, idque simili modo in fine ad superficiem CD fieri queat. Ex quo perspicuum est in genere tres variationes in calculum introduci debere, vt eas tam pro initio quam pro fine curuae inuestigandae ad superficies terminatrices transferre liceat, quippe quarum indoles in vtroque termino relationem mutuam inter variationes determinabit. Fig. 2.

Scholion 2.

19. Quemadmodum hic tres variabiles sumus contemplati quarum relatio duabus aequationibus determinatur, ita etiam calculus variabilium ad quatuor pluresue extendi potest, siquidem relatio per tot aequationes exprimatur vt per vnicam variabilem reliquae omnes determinationem suam nanciscantur, etiamsi huius casus illustratio non amplius

ex

ex Geometria tribus tantum dimenſionibus incluſa peti queat; niſi forte tempus in ſubſidium vocare velimus, fluuium continuum a ſuperficie A B ad ſuperficiem CD profluentem ſed temporis lapſu iugiter immutatum conſiderantes ita vt tum etiam temporis momentum ſit aſſignandum quo quaepiam fluuii vena a ſuperficie AB ad ſuperficiem CD porrecta maximi vel minimi proprietate quadam ſit praedita. Ad quas variabiles ſi inſuper celeritatis mutabilitatem adiiciamus, haec maiori variationum numero illuſtrando inſeruire poterunt. Imprimis autem hinc intelligitur, etiamſi omnes variabiles per vnicam determinari aſſumantur, rationem inveſtigationis tamen ab ea vbi duae tantum variabiles admittuntur, maxime diſcrepare, propterea quod ſingulis ſuae variationes a reliquis non pendentes tribui debent; neque enim inde, quod inter variabiles ipſas certa quaedam relatio agnoſcitur, ideo quoque earum variationes vlli relationi adſtrictae ſunt cenſendae. Veluti ex caſu ante allato manifeſtum eſt, vbi curua inter binas ſuperficies A B et CD porrecta et certa maximi minimiue proprietate praedita vtique ita eſt in ſe determinata, vt ſumta coordinatarum vna binae reliquae determinentur; nihilo vero minus curuae variatae omnes quae in omnes plagas ab illa deflectere poſſunt, pro ſingulis coordinatis recipiunt variationes neutiquam a ſe inuicem pendentes, ſolo initio ac fine excepto, vbi eas ad datas ſuperficies accommodari oportet.

Defini-

Definitio 4.

30. *Relatio inter ternas variabiles vnica aequatione definita vt vna earum aequetur functioni binarum reliquarum, variari dicitur, si vel vna vel omnes tres illae variabiles particulis infinite paruis augeantur, quae earum variationes vocantur.*

Explicatio.

31. Quoniam hic relatio inter ternas variabiles vnica aequatione definiri ponitur, duabus pro arbitrio sumtis tertia demum determinatur, ita vt pro functione duarum variabilium sit habenda. Ea ergo relatione non quaedam linea curua, si rem ad figuras transferre velimus, indicatur, sed tota quaedam superficies, cuius natura aequatione inter ternas coordinatas exprimitur, ex quo intelligitur, eadem relatione variata aliam superficiem ab illa infinite parum diffidentem repraesentari, quae variatio ita latissime patere debet, vt variatio vel tantum ad quampiam superficiei portionem restringi vel per totam extendi possit. Prout igitur cum quouis superficiei datae puncto aliud punctum superficiei variatae illi quidem proximum comparatur, fieri potest, vt non solum trium coordinatarum vna sed etiam duae vel adeo omnes tres varientur; vnde quo tractatio in omni amplitudine instituatur, conueniet statim singulis coordinatis suas tribui variationes, quas propterea ita comparatas esse oportet,

tet, vt tanquam functiones binarum variabilium spectari possint, cum binis demum determinatis superficiei punctum determinetur.

Coroll. 1.

22. Si igitur tres variabiles seu coordinatae sint x, y et z, quemadmodum ex relatione binis x et y pro lubitu valores tribuere licet, vnde z valorem determinatum obtineat, itidem variatio ipsius z ab vtraque illarum x et y pendere censenda est, quandoquidem siue alterutra siue ambae mutentur, alia variatio ipsius z resultare debet.

Coroll. 2.

23. Quod hic de variatione vnius z obseruatum est perinde de binis reliquis est intelligendum, ita vt singularum variationes sint tanquam functiones binarum variabilium considerandae; quoniam vero inter x et y et z aequatio datur, perinde est, quarumnam binarum functiones concipiantur, quia functio ipsarum y et z per aequationem ad functionem ipsarum x et y reuocari potest, si scilicet loco z suus valor per x et y expressus substituatur.

Scholion 1.

24. Hac variationum institutione erit vtendum, si superficies fuerit inuestiganda, quae maximi minimiue quapiam proprietate sit praedita, quandoqui-

doquidem calculum tum ita instrui oportet, vt eadem proprietas in superficies illi proximas ac variatas aeque competat. Deinde cum in curuis maximi minimiue proprietate praeditis amborum terminorum ratio praescribi soleat, vt vel in datis punctis, vel ad datas lineas curuas, vel adeo superficies terminentur, similis conditio hic est admittenda, vt superficies quaerenda circumquaque definiatur, vel data quadam superficie circumscribatur; cuius posterioris conditionis vt ratio haberi possit, omnino necesse est; vt omnibus tribus coordinatis variationes generalissimae a se inuicem neutiquam pendentes tribuantur, quo eae deinceps in extrema ora ad naturam superficiei, terminantis accommodari queant. Hic quidem fatendum est methodum maximorum et minimorum vix adhuc ad huiusmodi inuestigationes esse promotam tantasque difficultates hic occurrere, ad quas superandas multo maiora Analyseos incrementa requiri videntur. Verum ob hanc ipsam caussam eo magis erit enitendum vt principia huius methodi, quae calculo variationum continentur, solide stabiliantur, simulque clare ad distincte proponantur.

Scholion 2.

25. Vix opus esse arbitror hic animaduertere, istum calculum simili modo ad plures tribus variabiles amplificari posse, etiamsi quaestiones geometricae non amplius dilucidationem suppeditent; ipsa enim Analysis non vti Geometria certo dimensionum

num numero limitari eſt cenſenda. Quando autem plures variabiles conſiderantur, ante omnia perpendi conuenit, vtrum earum relatio mutua vnica tantum aequatione exprimatur, an pluribus? quae tot eſſe poſſunt, vt multitudo vnitate tantum a numero variabilium deficiat, quo caſu omnes tanquam functiones vnius ſpectare licet. Sin autem paucioribus aequationibus conſtet relatio, ſingulae variabiles erunt functiones duarum pluriumue variabilium, et quolibet quoque caſu variationes ſingulis tributae tanquam functiones totidem variabilium tractari debent, ſiquidem hunc calculum generaliſſime expedire velimus.

Definitio 5.

26. *Calculus variationum eſt methodus inueniendi variationem, quam recipit expreſſio ex quotcunque variabilibus vtcunque conflata, dum variabilibus vel omnibus vel aliquibus variationes tribuuntur.*

Explicatio.

27. In hac definitione nulla fit mentio relationis, quam hactenus inter variabiles dari aſſumſimus, cum enim hic calculus potiſſimum in hac ipſa relatione inueſtiganda ſit occupatus, quae ſcilicet maximi minimiue proprietate ſit praedita quamdiu ea adhuc eſt incognita, eius rationem in calculo neutiquam habere licet, ſed potius eum ita tractari conuenit, quaſi variabiles nulla plane relatione

tione inter se essent connexae. Calculum igitur ita instrui conuenit, vt si singulis variabilibus, quae in calculum ingrediuntur, variationes tribuantur quaecunque omnis generis expressionum, quae vtcunque ex iis fuerint conflatae, variationes inde oriundae inuestigari doceantur, quibus in genere inuentis tum demum eiusmodi quaestiones euoluendae occurrunt, qualem relationem inter variabiles statui oporteat vt variatio illa inuenta sit vel nulla, vti in inuestigatione maximorum seu minimorum vsu venit, vel alio certo quodam modo sit comparata, prout natura quaestionum exegerit. Hoc modo si istius calculi praecepta tradantur, nihil impedit, quo minus etiam eiusmodi quaestiones tractentur, in quibus statim relatio quaedam inter variabiles tanquam data assumitur ac certae cuiusdam expressionis ex iis formatae variatio ex variabilium variationibus nata desideratur. Ex quo intelligitur, hunc calculum ad quaestiones plurimas diuersissimi generis accommodari posse.

Coroll. 1.

28. Quaestiones ergo in hoc calculo tractandae huc redeunt, vt proposita expressione quacunque ex quotcunque variabilibus vtcunque conflata, eius incrementum definiatur, si singulae variabiles suis variationibus augeantur.

Coroll. 2.

29. Similis igitur omnino eſt calculus variationum calculo differentiali, dum in vtroque variabilibus incrementa infinite parua tribuuntur. Quatenus autem vti iam obſeruauimus, variationes a differentialibus diſcrepant, adeoque ſimul cum iis conſiſtere poſſunt, eatenus ſummum diſcrimen inter vtrumque calculum eſt agnoſcendum.

Scholion.

30. Ex obſeruationibus ſupra allatis diſcrimen hoc maxime fit manifeſtum, vbi enim calculus refertur ad lineam curuam, quam cum alia ſibi proxima comparari oportet, per differentialia a puncto quouis curuae ad alia puncta eiusdem curuae progredimur, quando autem ab hac curua ad alteram ſibi proximam tranſilimus, tranſitus quatenus eſt infinite paruus, fit per variationes. Idem euenit in ſuperficiebus ad alias ſibi proximas relatis, vbi differentialia in eadem ſuperficie concipiuntur, variationibus vero ab vna in alteram tranſilitur. Eadem omnino eſt ratio, ſi res analytice conſideretur ſine vllo reſpectu ad figuras geometricas, vbi ſemper variationes quantitatum variabilium a ſuis differentialibus ſollicite diſtingui oportet quem in finem variationes ſigno diuerſo indicari conueniet.

Hypothesis.

31. *Variationem cuiusque quantitatis variabilis littera δ eidem quantitati praefixa in posterum designabimus, ita vt* δx, δy, δz *designent variationes quantitatum* x, y, z, *ac si* V *fuerit expressio quaecunque ex iis conflata eius variatio hoc modo* δV *nobis indicabitur.*

Coroll. 1.

32. Significat ergo δx incrementum illud infinite paruum quo quantitas x augeri concipitur, vt eiusdem valor variatus prodeat ex quo vicissim intelligitur valorem variatum ipsius x fore $x+\delta x$.

Coroll. 2.

33. Quatenus ergo expressio V ex variabilibus x, y et z conflatur, si earum loco scribantur valores variati $x+\delta x$, $y+\delta y$ et $z+\delta z$, atque a valore hoc modo pro V resultante subtrahatur ipsa V residuum erit variatio δV.

Coroll. 3.

34. Hactenus ergo omnia perinde se habent atque in calculo differentiali, ac si V fuerit functio quaecunque ipsarum x, y et z, sumto eius differentiali more solito tantum vbique loco d scribatur δ, et habebitur eius variatio δV.

Scho-

Scholion 1.

35. Quoties ergo V eſt functio quaecunque quantitatum variabilium x, y, z eius variatio iisdem regulis inde elicitur ac differentiale eius, ex quo calculus variationum prorſus cum calculo differentiali congruere videri poſſet, cum ſola ſigni diverſitas leuis ſit momenti. Verum probe perpendendum eſt hic non omnes quantitates, quarum variationes requiruntur, in genere functionum comprehendi poſſe; quamobrem etiam in definitione vocabulo *expreſſionis* ſum vſus, cui longe ampliorem ſignificatum attribuo. Quatenus enim ad relationem mutuam variabilium reſpicere non licet, quia eſt incognita, eatenus eiusmodi expreſſiones ſeu formulae in quas variabilium differentialia atque etiam integralia ingrediuntur, non amplius tanquam merae functiones variabilium ſpectari poſſunt, ac formularum tam differentialium quam integralium variatio peculiaria praecepta poſtulat; ſicque totum negotium huc redit, vt quemadmodum formularum vtriusque generis variationes inueſtigari conueniat, doceamus, ex quo tractatio noſtra euadit bipartita.

Scholion 2.

36. In ipſa autem tractatione maximum exoritur diſcrimen ex numero variabilium, qui ſi binarium ſuperet, vix adhuc perſpicitur, quomodo calculus ſit expediendus. Cum enim pluribus intro-

troductis variabilibus, etiam differentialium consideratio longe aliter expendatur, dum plerumque binarum tantum differentialia ita inter se comparari solent, quasi reliquae variabiles manerent constantes, similis quoque ratio in variationibus erit habenda in quo etiamnunc tantae difficultates occurrunt, vt vix pateat quomodo eas superare liceat; ante omnia certe prima huius calculi principia accuratissime euolui erit necesse, vt ex intima rei natura calculi praecepta repetantur, in quo plerumque summae difficultates offendi solent. Primum igitur hunc calculum ad duas tantum variabiles accommodatum, quemadmodum is quidem adhuc tractari est solitus, explicare conabor, variationes tam formularum differentialium quam integralium inuestigaturus, tum vero si quid lucis ex ipsa hac tractatione affulserit quoque ad tres pluresue variabiles contemplandas progrediar.

CAPVT II.

DE VARIATIONE FORMVLARVM DIFFERENTIALIVM DVAS VARIABILES INVOLVENTIVM.

Theorema 1.

37.

Variatio differentialis semper aequalis est differentiali variationis, seu est $\delta d V = d \delta V$, *quaecunque fuerit quantitas* V, *quae dum per differentialia crescit, etiam variationem recipit.*

Demonstratio.

Quantitas variabilis V spectari potest, tanquam applicata curuae cuiuspiam, quae suis differentialibus per candem curuam progrediatur, suis variationibus vero in aliam curuam illi proximam transiliat. Dum autem in eiusdem curuae punctum proximum promouetur, fit eius valor $= V + dV$ qui fit $= V'$ ideoque $dV = V' - V$; ex quo variatio ipsius dV hoc est δdV erit $= \delta V' - \delta V$. Verum $\delta V'$ est valor proximus, in quem δV suo differentiali auctum abit, ita vt sit $\delta V' = \delta V + d\delta V$ seu $\delta V' - \delta V = d\delta V$ vnde euidens est fore $\delta dV = d\delta V$, seu variationem differentialis esse aequalem differentiali variationis, prorsus vti Theorema affirmat.

Coroll. 1.

38. Hinc variatio differentialis secundi ddV ita definitur vt sit $\delta ddV = d\delta . dV$, at cum sit $\delta dV = d\delta V$ aequalitas erit inter has formulas:

$$\delta ddV = d\delta dV = dd\delta V.$$

Coroll. 2.

39. Eodem modo pro differentialibus tertii ordinis erit

$$\delta d^3V = d\delta ddV = dd\delta dV = d^3\delta V$$

et pro differentialibus quarti ordinis variatio ita se habebit vt sit

$$\delta d^4V = d\delta d^3V = dd\delta ddV = d^3\delta dV = d^4\delta V$$

similique modo pro altioribus gradibus.

Coroll. 3.

40. Si igitur variatio desideretur differentialis cuiuscunque gradus signum variationis δ vbicunque libuerit inter signa differentiationis d inseri potest; in vltimo autem loco positum declarat, variationem differentialis cuiusuis gradus aequalem esse differentiali eiusdem gradus ipsius variationis.

 Coroll. 4.

Coroll. 4.

41. Cum igitur sit $\delta d^n V = d^n \delta V$, res semper eo reducitur, vt variationis quantitatis V seu ipsius δV differentialia cuiusque gradus capi possint; atque in hac reductione praecipua vis huius noui calculi est constituenda.

Scholion 1.

42. Vis demonstrationis in hoc potissimum est sita, quod δV abeat in $\delta V'$, si quantitas V suo differentiali increscit, quod quidem ex natura differentialium per se est manifestum; interim tamen iuuabit id per Geometriam illustrasse. Pro curua
Fig. 3. quacunque EF sint coordinatae $AX = x$ et $XY = y$, in qua si per interuallum infinite paruum YY' progrediamur erit in differentialibus

$$AX' = x + dx \text{ et } X'Y' = y + dy,$$

ideoque

$$dx = AX' - AX \text{ et } dy = X'Y' - XY.$$

Nunc concipiamus aliam curuam ef illi proximam, cuius puncta y et y' cum illius punctis Y et Y' comparentur, ad quae propterea per variationes transitus fiat; ac sumtis simili modo coordinatis erit

$$Ax = x + \delta x \text{ et } xy = y + \delta y,$$

ideoque

$$\delta x = Ax - AX \text{ et } \delta y = xy - XY,$$

tum

tum vero erit

$$Ax' = x + dx + \delta(x + dx) \text{ et } x'y' = y + dy + \delta(y + dy)$$

quatenus a puncto Y' per variationem in punctum y' transfilimus. Verum ad idem punctum y' quoque ex puncto y per differentiationem peruenimus vnde colligitur:

$$Ax' = x + \delta x + d(x + \delta x) \text{ et } x'y' = y + \delta y + d(y + \delta y).$$

His iam valoribus cum illis collatis prodit

$$x + dx + \delta x + \delta dx = x + \delta x + dx + d\delta x \text{ et}$$

$$y + dy + \delta y + \delta dy = y + \delta y + dy + d\delta y$$

vnde manifesto sequitur fore:

$$\delta dx = d\delta x \text{ et } \delta dy = d\delta y.$$

Quae si attentius consideremus, principium, cui demonstratio innititur, huc redire comperimus, vt si quantitas variabilis primo per differentiationem deinde vero per variationem proferatur, idem proveniat, ac si ordine inuerso primo per variationem tum vero per differentiationem promoueretur. Veluti in figura ex puncto Y primo per differentiationem peruenitur in Y', hinc vero per variationem in y': inuerso autem ordine primum ex puncto Y per variationem peruenitur in y, hinc vero per differentiationem in punctum y', idem quod ante.

 Scho-

Scholion 2.

43. Theorema hoc latissime patet, neque enim ad casum duarum variabilium tantum restringitur, sed veritati est etiam consentaneum, quotcunque variabiles in calculum ingrediantur, quandoquidem in demonstratione solius illius variabilis cuius tam differentiale quam variatio consideratur, ratio habetur sine vllo respectu ad reliquas variabiles. Ne autem hic vlli dubio locus relinquatur, Fig. 4. consideremus superficiem quamcunque, cuius punctum quoduis Z per coordinatas ternas

$$AX = x,\; XY = y \text{ et } YZ = z$$

definiatur, a quo si ad aliud punctum proximum Z' in eadem superficie progrediamur, hae coordinatae suis differentialibus increscent. Tum vero aliam quamcunque superficiem concipiamus proximam cuius puncta z et z' cum illis Z et Z' conferantur, quod fit per variationem. His positis perspicuum est duplici modo ad punctum z' perueniri posse, altero per variationem ex puncto Z', altero per differentiale ex puncto z, sicque fore:

$$Ax' = AX' + \delta.\, AX' = Ax + d.\, Ax$$

$$x'y' = X'Y' + \delta.\, X'Y' = xy + d.\, xy$$

$$y'z' = Y'Z' + \delta.\, Y'Z' = yz + d.\, yz$$

quod etiam de omnibus aliis quantitatibus variabilibus ad haec puncta referendis valet. Hinc autem luculenter sequitur fore

$$\delta dx = d\delta x,\; \delta dy = d\delta y,\; \delta dz = d\delta z.$$

Scho-

Scholion 3.

44. Memorabile prorſus eſt, quod caſu differentialium altioris ordinis ſignum variationis δ pro lubitu inter ſigna differentiationis d inſcribi poſſit, atque hinc intelligere licet hanc permutabilitatem locum quoque eſſe habituram, etiamſi ſignum variationis δ perinde ac differentiationis d aliquoties repetatur, quod fortaſſe in aliis ſpeculationibus vſu venire poſſet. Verum in praeſenti inſtituto repetitio variationis δ nullo modo locum habere poteſt, quoniam lineam vel ſuperficiem tantum cum vnica alia ſibi proxima comparamus, etſi enim haec generaliſſime conſideratur, vt omnes poſſibiles itidem proximas in ſe complectatur, tamen tanquam vnica ſpectatur, neque poſtquam e principali in proximam tranſiliuerimus, nouus tranſitus in aliam conceditur. Hinc ergo eiusmodi ſpeculationes, quibus variationum variationes eſſent quaerendae, omnino excluduntur. Viciſſim autem hic variationum differentialia, cuiusque ordinis hic admitti debent, et cum in formulis differentialibus quae quidem ſignificatum habent finitum, ratio differentialium tantum ſpectetur, quae ſi binae variabiles ſint x et y, huiusmodi poſitionibus

$$dy = p\,dx, \quad dp = q\,dx, \quad dq = r\,dx \text{ etc.}$$

ad formas finitas reuocari ſolent, harum quantitatum p, q, r, etc. variationes potiſſimum aſſignari neceſſe eſt.

Problema

Problema I.

45. Datis binarum variabilium x et y variationibus δx et δy, formulae differentialis $p = \frac{dy}{dx}$ variationem definire.

Solutio.

Cum sit $\delta dy = d\delta y$ et $\delta dx = d\delta x$, variatio quaesita δp per notas differentiationis regulas reperitur, dummodo loco signi differentiationis d scribatur, signum variationis δ, vnde cum oriatur

$$\delta p = \frac{dx\delta dy - dy\delta dx}{dx^2},$$

erit per conuersionem ante demonstratam:

$$\delta p = \frac{dx\delta dy - dyd\delta x}{dx^2}$$

vbi cum δx et δy sint variationes ipsarum x et y, hincque $\delta x + d\delta x$ et $\delta y + d\delta y$ variationes ipsarum $x + dx$ et $y + dy$ notandum est fore vti iam obseruauimus:

$$d\delta x = \delta(x + dx) - \delta x \text{ et } d\delta y = \delta(y + dy) - \delta y.$$

Idem inuenitur ex primis principiis cum enim valor ipsius variatus sit $p + \delta p$ isque prodeat si loco x et y earum valores variati, qui sunt $x + \delta x$ et $y + \delta y$ substituantur erit

$$p + \delta p = \frac{d(y + \delta y)}{d(x + \delta x)} = \frac{dy + d\delta y}{dx + d\delta x},$$

vnde

vnde ob $p = \frac{dy}{dx}$ fit

$$\delta p = \delta . \frac{dy}{dx} = \frac{dy + d\delta y}{dx + d\delta x} - \frac{dy}{dx} = \frac{dx d\delta y - dy d\delta x}{dx^2}$$

quoniam in denominatore particula $dx d\delta x$ prae dx^2 euanefcit.

Coroll. 1.

46. Si dum per differentialia progredimur, variabiles x et y continuo auctas defignemus, per x', x'', x''' etc. y', y'', y''' etc. vt fit

$$x' = x + dx \text{ et } y' = y + dy, \text{ erit}$$

$$d\delta x = \delta x' - \delta x \text{ et } d\delta y = \delta y' - \delta y$$

hincque

$$\delta p = \delta . \frac{dy}{dx} = \frac{dx(\delta y' - \delta y) - dy(\delta x' - \delta x)}{dx^2}.$$

Coroll. 2.

47. Quoniam variationes ambarum variabilium x et y neutiquam a fe inuicem pendent, fed prorfus arbitrio noftro relinquuntur, fi ipfi x nullas tribuamus variationes vt fit

$$\delta x = 0 \text{ et } \delta x' = 0, \text{ erit}$$

$$\delta p = \frac{d\delta y}{dx} = \frac{\delta y' - \delta y}{dx}.$$

Coroll. 3.

48. Si praeterea vnicae variabili y variationem δy tribuamus, vt fit $\delta y' = 0$ erit $\delta p = -\frac{dy}{dx}$, quae hypothefis minime naturae refragatur, quia

curuam proximam ita cum principali congruentem assumi licet, vt in vnico tantum puncto ab ea discrepet.

Scholion.

49. Vulgo in solutione problematum isoperimetricorum aliorumque ad id genus pertinentium, curua variata ita congruens statui solet, vt tantum in vno quasi elemento discrepet. Ita si quaerenda sit curua

Fig. 5. E F certa quadam maximi minimiue proprietate gaudens, vnicum punctum Y in locum proximum y transferri solet vt curua variata $EMyY'F$ tantum in interuallo minimo MY' a quaesita deflectat ita vt positis

$$AX = x \text{ et } XY = y$$

sit pro variata curua

$$Ax = x + \delta x \text{ et } xy = y + \delta y, \text{ seu}$$

$$\delta x = Ax - AX \text{ et } \delta y = xy - XY,$$

pro sequentibus vero punctis, ad quae differentialia ducunt sit vbique

$$\delta x' = 0, \ \delta y' = 0, \ \delta x'' = 0, \ \delta y'' = 0, \text{ etc.}$$

itemque pro antecedentibus. Quin etiam ad calculi commodum variatio $Xx = \delta x$ nulla sumi solet, vt omnis variatio ad solum elementum δy perducatur, quo casu vtique habebitur $\delta p = -\frac{\delta y}{dx}$, haecque vnica variatio vtique sufficit ad problemata huius generis quae quidem fuerint tractata, resoluenda. Verum si, vti

vti hic instituimus, haec problemata latius extendimus, vt curua quaesita circa initium et finem certas determinationes recipere queat, vtique necessarium est calculum variationum quam generalissime absolvere, atque in omnibus curuae punctis variationes indefinitas coordinatis tribuere. Quod etiam maxime est necessarium si huiusmodi inuestigationes ad lineas curuas non continuas accommodare velimus.

Problema 2.

50. Datis binarum variabilium x et y variationibus δx et δy, si ponatur $dy = p dx$ et $dp = q dx$, inuenire variationem quantitatis q, seu valorem ipsius δq.

Solutio.

Cum sit $q = \frac{dp}{dx}$, erit pro valore variato

$$q + \delta q = \frac{d(p + \delta p)}{d(x + \delta x)} = \frac{dp + d\delta p}{dx + d\delta x},$$

vnde auferendo quantitatem p relinquitur

$$\delta q = \frac{dx\, d\delta p - dp\, d\delta x}{dx^2},$$

quae variatio ergo etiam ex differentiatione formulae $q = \frac{dp}{dx}$ resultat, si more consueto differentiatio instituatur, loco vero signi differentialis d scribatur signum variationis δ; vbi quidem meminisse iuuabit esse

$$\delta dx = d\delta x \text{ et } \delta dp = d\delta p.$$

Supra autem inuenimus ob $p=\frac{dy}{dx}$ esse

$$\delta p=\frac{dx\,d\delta y-dy\,d\delta x}{dx^2},$$

vnde porro per consuetam differentiationem valor ipsius $d\delta p$ scilicet differentiale ipsius δp colligitur.

Coroll. 1.

51. Cum sit $\frac{dy}{dx}=p$ et $\frac{dp}{dx}=q$, erit primo:

$$\delta p=\frac{d\delta y}{dx}-\frac{p\,d\delta x}{dx}$$

tum vero

$$\delta q=\frac{d\delta p}{dx}-\frac{q\,d\delta x}{dx}.$$

Pro vsu autem futuro praestat hic particulam $d\delta p$ relinqui, quam eius valorem ex praecedente formula erui.

Coroll. 2.

52. Interim tamen cum prior per differentiationem det

$$d\delta p=\frac{dd\delta y}{dx}-\frac{ddx\,d\delta y}{dx^2}-\frac{p\,dd\delta x}{dx}-q\,d\delta x+\frac{p\,ddx\,d\delta x}{dx^2}$$

hoc valore substituto prodit

$$\delta q=\frac{dd\delta y}{dx^2}-\frac{ddx\,d\delta y}{dx^3}-\frac{p\,dd\delta x}{dx^2}-\frac{2q\,d\delta x}{dx}+\frac{p\,ddx\,d\delta x}{dx^3}.$$

Coroll. 3.

53. Quod si soli variabili y variationes tribuantur, vt particulae δx et quae inde deriuantur euane-

euanescant, habebimus

$$\delta p = \frac{d\delta y}{dx} \text{ et } \delta q = \frac{d\delta p}{dx^2} = \frac{dd\delta y}{dx^2} - \frac{ddx\,d\delta y}{dx^3}$$

ac si differentiale dx constans accipiatur, erit $\delta q = \frac{dd\delta y}{dx^2}$.

Scholion 1.

54. Quo haec facilius intelligantur, consideremus in curua EF, per relationem inter variabiles $AX = x$ et $XY = y$ plura puncta Y, Y', Y'' etc. secundum differentialia continuo promota, vt sit Fig. 5.

$$AX = x;\ AX' = x + dx;\ AX'' = x + 2dx + ddx;\ AX''' = x + 3dx + 3ddx + d^3x$$

$$XY = y;\ X'Y' = y + dy;\ X''X'' = y + 2dy + ddy;\ X'''Y''' = y + 3dy + 3ddy + d^3y$$

quae differentialia cuiusque ordinis indicantes ita breuitatis gratia repraesententur:

$$AX = x;\ AX' = x';\ AX'' = x'';\ AX''' = x''' \text{ etc.}$$

$$XY = y;\ X'Y' = y';\ X''Y'' = y'';\ X'''Y''' = y''' \text{ etc.}$$

quibus singulis suae variationes nullo modo a se inuicem pendentes tribui concipiantur, ita vt omnes istae variationes:

$$\delta x,\ \delta x',\ \delta x'',\ \delta x''' \text{ etc.}$$

$$\delta y,\ \delta y',\ \delta y'',\ \delta y''' \text{ etc.}$$

a lubitu nostro pendentes tanquam cognitae spectari queant. His iam ita constitutis differentialia cuiusque ordinis variationum in hunc modum repraesentabuntur vt sit

$$d\delta x = \delta x' - \delta x\ ;\quad dd\delta x = \delta x'' - 2\delta x' + \delta x\ ;$$
$$d^3\delta x = \delta x''' - 3\delta x'' + 3\delta x' - \delta x$$
$$d\delta y = \delta y' - \delta y\ ;\quad dd\delta y = \delta y'' - 2\delta y' + \delta y\ ;$$
$$d^3\delta y = \delta y''' - 3\delta y'' + 3\delta y' - \delta y.$$

Quodsi iam vnicum punctum curuae Y variari sumamus, erit

$$d\delta x = -\delta x\ ;\ dd\delta x = +\delta x\ ;\ d^3\delta x = -\delta x \text{ etc.}$$
$$d\delta y = -\delta y\ ;\ dd\delta y = +\delta y\ ;\ d^3\delta y = -\delta y \text{ etc.}$$

hincque

$$\delta p = -\frac{\delta y}{dx} + \frac{p\,\delta x}{dx} \text{ et}$$
$$\delta q = \frac{\delta y}{dx^2} + \frac{ddx\,\delta y}{dx^3} - \frac{p\,\delta x}{dx^2} + \frac{2q\,\delta x}{dx} - \frac{p\,ddx\,\delta x}{dx^3}$$

vbi omissis partibus reliquarum respectu euanescentibus, erit

$$\delta q = \delta y . \frac{1}{dx^2} - \delta x . \frac{p}{dx^2}.$$

Denique si soli applicatae $XY = y$ variatio tribuatur, habebitur:

$$\delta p = -\frac{1}{dx}.\delta y \text{ et } \delta q = \frac{1}{dx^2}.\delta y.$$

Scholion 2.

55. Hinc patet si in vnico curuae puncto variatio statuatur, insigniter contra recepta differentialium principia impingi, cum variationum differentialia superiora neutiquam prae inferioribus euanescant

nefcant fed iugiter eundem valorem retineant, atque adeo variationes quantitatum p et q in infinitum excrefcant, fiquidem infinite parua δx et δy ex eodem ordine quo differentialia dx et dy affumantur. Quin etiam hinc in calculo maxime cavendum eft ne in enormes errores praecipitemur, cum calculi praecepta, legi continuitatis innitantur, qua lineae curuae continuo puncti fluxu defcribi concipiuntur, ita vt in earum curuatura nusquam faltus agnofcatur. Quodfi autem vnicum curuae punctum Y in y diducatur reliquo curuae tractu praeter bina quafi elementa My et yY' inuariato relicto euidens eft curuaturae ingentem irregularitatem induci, cum vulgares calculi regulae non amplius applicari queant. Cui incommodo vt occurramus tutiffimum erit remedium, vt fingulis curuae punctis mente faltem variationes tribuantur, quae continuitatis quapiam lege contineantur, neque ante irregularitas in calculo admittatur, quam omnes differentiationes et integrationes fuerint peractae; hocque modo faltem fpecies continuitatis in calculo retineatur. Quamuis ergo variationum differentialia Fig. 5.

$$d\delta y,\ dd\delta y,\ d^3\delta y \text{ etc. item}$$
$$d\delta x,\ dd\delta x,\ d^3\delta x \text{ etc.}$$

forte in facta hypothefi ad fimplices variationes revocare liceat, tamen expedit illas formas in calculo retineri ad easque fequentes integrationes accommodari, atque huc etiam redeunt operationes, quas olim

olim, cum idem argumentum de inueniendis curvis maximi minimiue proprietate praeditis tractassem, expedire docueram.

Problema 3.

56. Datis binarum variabilium x et y variationibus δx et δy, rationum inter differentialia cuiuscunque gradus variationes inuestigare.

Solutio.

Quaestio huc redit vt positis continuo

$$dy = p\,dx,\ dp = q\,dx,\ dq = r\,dx,\ dr = s\,dx \text{ etc.}$$

quantitatum p, q, r, s etc. variationes assignentur, cum ad has quantitates omnes differentialium cuiuscunque ordinis rationes quae quidem finitis valoribus continentur reducantur. Ac de harum quidem duabus primis p et q iam vidimus esse

$$\delta p = \frac{d\delta y}{dx} - \frac{p\,d\delta x}{dx} \text{ et } \delta q = \frac{d\delta p}{dx} - \frac{q\,d\delta x}{dx}.$$

Quoniam igitur porro est

$$r = \frac{dq}{dx} \text{ et } s = \frac{dr}{dx} \text{ etc.}$$

harum variationes simili modo per differentiationis regulas inueniuntur:

$$\delta r = \frac{d\delta q}{dx} - \frac{r\,d\delta x}{dx};\ \delta s = \frac{d\delta r}{dx} - \frac{s\,d\delta x}{dx} \text{ etc.}$$

vbi si lubuerit loco $d\delta p$, $d\delta q$, $d\delta r$, etc. differentialia variationum δp, δq, δr etc. ante inuentarum

tarum ſubſtitui poſſunt. Hoc autem non ſolum in formulas nimis prolixas induceret, ſed etiam vti ex ſequentibus patebit ne quidem eſt neceſſarium, cum hinc multo facilius omnes reductiones, quibus opus erit, inſtitui queant.

Coroll. 1.

57. Si ſoli variabili y variationes tribuantur, ſeu manentibus abſciſſis x tantum applicatae y ſuis variationibus augeantur, habebimus:

$$\delta p = \frac{d\delta y}{dx}; \quad \delta q = \frac{d\delta p}{dx}; \quad \delta r = \frac{d\delta q}{dx}; \quad \delta s = \frac{d\delta r}{dx}.$$

Coroll. 2.

58. Quodſi praeterea omnia ipſius x incrementa dx aequalia capiantur, ſeu elementum dx conſtans ſtatuatur, ſubſtitutis differentialibus praecedentium formularum in ſequentibus obtinebitur:

$$\delta p = \frac{d\delta y}{dx}; \quad \delta q = \frac{dd\delta y}{dx^2}; \quad \delta r = \frac{d^3\delta y}{dx^3}; \quad \delta s = \frac{d^4\delta y}{dx^4} \text{ etc.}$$

Coroll. 3.

59. Si ſolis abſciſſis x variationes tribuantur, vt variatio δy cum omnibus deriuatis euaneſcat, ſimulque elementum dx conſtans capiatur, ſingulae hae variationes ita ſe habebunt:

$$\delta p = \frac{-p\,d\delta x}{dx}; \quad \delta q = \frac{-p\,dd\delta x}{dx^2} - \frac{2q\,d\delta x}{dx}$$

$$\delta r = \frac{-p\,d^3\delta x}{dx^3} - \frac{3q\,dd\delta x}{dx^2} - \frac{3r\,d\delta x}{dx}$$

$$\delta s = \frac{-p\,d^4\delta x}{dx^4} - \frac{4q\,d^3\delta x}{dx^3} - \frac{6r\,dd\delta x}{dx^2} - \frac{4s\,d\delta x}{dx}$$

etc.

Coroll. 4.

60. Etiamsi ergo hoc casu elementum dx constans accipiatur, tamen hic occurrunt differentialia cuiusque ordinis variationis δx, cuius rei ratio est, quod variationes valorum ipsius x continuo vlterius promotorum x', x'' etc. neutiquam a differentialibus pendere statuuntur.

Scholion.

61. Quando autem placuerit soli variabili x variationes tribuere, tum omnino praestat variabiles x et y inter se permutari, atque huiusmodi potius positionibus vti

$$dx = p\,dy, \quad dp = q\,dy, \quad dq = r\,dy \text{ etc.}$$

quibus species differentialium tollatur, tum vero sumto elemento dy constante similes formulae simpliciores pro variationibus quantitatum p, q, r etc. reperiuntur, atque Coroll. 2. Ceterum quo calculus ad omnes casus accommodari queat, semper expedit vtrique variabili suas variationes tribui, etsi enim tum formae multo perplexiores prodeant praecipue si euoluantur, tamen calculum prosequendo tam egregia se offerunt compendia vt in fine calculus vix fiat operosior, neque huius prolixitatis taedeat. Ad problemata ergo magis generalia ad hoc caput pertinentia progrediamur.

Problema

Problema 4.

62. Datis duarum variabilium x et y variationibus δx et δy, formulae cuiuscunque finitae V tam ex illis variabilibus ipsis quam earum differentialibus cuiuscunque ordinis conflatae variationem inuenire.

Solutio.

Cum V sit quantitas valorem habens finitum, ponendo

$$dy = p\,dx, \quad dp = q\,dx, \quad dq = r\,dx, \quad dr = s\,dx \text{ etc.}$$

differentialia inde tollentur, prodibitque pro V functio ex quantitatibus finitis formata $x, y\, p, q, r, s$ etc. Quaecunque ergo sit ratio compositionis eius differentiale semper huiusmodi habebit formam:

$$dV = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds \text{ etc.}$$

horum membrorum numero existente eo maiore, quo altiora differentialia ingrediuntur in V. Quodsi vero huius formulae V variatio δV fuerit indaganda, ea obtinetur si loco quantitatum variabilium x, y, p, q, r etc. eaedem suis variationibus auctae substituantur et a forma resultante ipsa quantitas V auferatur, ex quo intelligitur variationem ope consuetae differentiationis inueniri signo tantum differentialis d in signum variationis δ mutato. Quare cum differentiale supra iam sit exhibitum impetrabimus variationem quaesitam:

$$\delta V = M\,\delta x + N\,\delta y + P\,\delta p + Q\,\delta q + R\,\delta r + S\,\delta s \text{ etc.}$$

 quem-

quemadmodum autem variationes $\delta p, \delta q, \delta r, \delta s$ etc. per variationes ſumtas δx et δy determinentur iam ſupra eſt oſtenſum.

Coroll. 1.

63. Si hic ſubſtituamus valores ante inuentos, obtinebimus variationem quaeſitam ita expreſſam

$$\delta V = M\delta x + N\delta y + \frac{1}{dx}(Pd\delta y + Qd\delta p + Rd\delta q + Sd\delta r + \text{etc.})$$
$$- \frac{d\delta x}{dx}(Pp + Qq + Rr + Ss + \text{etc.}).$$

Coroll. 2.

63. Si variabili x nulla plane tribuatur variatio, atque inſuper elementum dx conſtans accipiatur, tum quantitatis propoſitae V variatio ita prodibit expreſſa:

$$\delta V = N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} + \text{etc.}$$

Scholion.

65. In his formis ſaltem ſpecies homogeneitatis in differentialibus ſpectatur, ſiquidem δx et δy ad ordinem differentialium referantur, quod longe ſecus eueniret, ſi eo caſu quo vnicum curuae punctum variatur, ſtatim vellemus loco differentialium variationum valores ſupra (54.) exhibitos ſubſtituere, quo quippe pacto idea integrationis, qua hae formulae deinceps indigent excluderetur. Ceterum patet quomodo inuentio variationum ad conſuetam diffe-

differentiationem reuocetur, dum totum diſcrimen in hoc tantum eſt ſitum vt loco variationum δp, δq, δr etc. valores iam ante aſſignati, quos quidem ipſos quoque per conſuetam differentiationem elicuimus, ſubſtituantur. Conueniet autem hanc operationem aliquot exemplis illuſtrari quo clarius indoles totius huius tractationis percipiatur.

Exemplum 1.

66. *Formulae ſubtangentem exprimentis* $\frac{y\,dx}{dx}$ *variationem inuenire.*

Ob $dy = p\,dx$ haec formula fit $\frac{y}{p}$, vnde eius variatio $\frac{\delta y}{p} - \frac{y\,\delta p}{pp}$, vbi loco δp valore ſubſtituto fit ea:

$$\frac{\delta y}{p} - \frac{y\,d\,\delta y}{pp\,dx} + \frac{y\,d\,\delta x}{p\,dx} = \frac{dx}{dy}.\delta y - \frac{y\,dx}{dy^2}.d\,\delta y + \frac{y}{dy}d\,\delta x$$

quae poſtrema forma immediate ex differentiatione formulae propoſitae naſcitur.

Exemplum 2.

67. *Formulae ipſam tangentem exprimentis* $\frac{y\sqrt{(dx^2+dy^2)}}{dy}$ *variationem inuenire.*

Poſitio $dy = p\,dx$ praebet hanc formam finitam $\frac{y}{p}\sqrt{(1+pp)}$ vnde variatio quaeſita eſt:

$$\frac{\delta y}{p}\sqrt{(1+pp)} - \frac{y\,\delta p}{pp\sqrt{(1+pp)}},$$

quae transformatur in hanc

$$\frac{\sqrt{(dx^2+dy^2)}}{dy}\delta y - \frac{y\,dx}{dy^2\sqrt{(dx^2+dy^2)}}(dx\,d\,\delta y - dy\,d\,\delta x).$$

Exemplum 3.

68. *Formulae radium curuedinis exprimentis* $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$ *variationem definire.*

Posito $dy=pdx$ et $dp=qdx$ haec formula transit in hanc $\frac{(1+pp)^{\frac{3}{2}}}{q}$, cuius propterea variatio est

$$\frac{3p\delta p}{q}\sqrt{(1+pp)}-\frac{\delta q}{qq}(1+pp)^{\frac{3}{2}}$$

vbi quidem substitutioni valorum ante inuentorum non immoror.

Problema 5.

69. Datis duarum quantitatum variabilium x et y variationibus δx et δy formulae tam ex illis variabilibus quam earum differentialibus cuiuscunque ordinis conflatae siue fuerit infinita siue infinite parua, variationem inuestigare.

Solutio.

Positis vt hactenus $dy=pdx$, $dp=qdx$, $dq=rdx$ etc. formula semper reducetur ad huiusmodi formam Vdx^n, vbi V sit functio finita quantitatum x, y, p, q, r etc. exponens vero n siue positiuus siue negatiuus, ita vt priori casu formula sit infi-

infinite parua, posteriori vero infinite magna. Ponamus igitur differentiationem ordinariam dare

$$dV = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + \text{etc.}$$

vnde simul eius variatio habetur. Cum igitur formae propositae variatio sit:

$$n V\, dx^{n-1}\, d\delta x + dx^n\, \delta V$$

erit vtique haec variatio quam quaerimus:

$$n V\, dx^{n-1}\, d\delta x + dx^n (M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r\, \text{etc.})$$

vbi ex superioribus hos valores substitui oportet:

$$\delta p = \frac{d\delta y - p\, d\delta x}{dx}; \quad \delta q = \frac{d\delta p - q\, d\delta x}{dx}$$

$$\delta r = \frac{d\delta q - r\, d\delta x}{dx}; \quad \delta s = \frac{d\delta r - s\, d\delta x}{dx}$$

etc.

quae cum per se sint perspicua, nulla ampliori explicatione indigent; simulque hoc caput penitus absolutum videtur.

CAPVT

CAPVT III.

DE VARIATIONE FORMVLARVM INTEGRALIVM SIMPLICIVM DVAS VARIABILES INVOLVENTIVM.

Definitio 6.

70.

Formulam integralem simplicem hic appello, quae nulla alia integralia in se inuoluit, sed simpliciter integrale refert formulae differentialis, praeter binas variabiles quaecunque earum differentialia complectentis.

Coroll. I.

71. Si ergo x et y sint binae variabiles, formula integralis $\int W$ erit simplex, si expressio W praeter has variabiles tantum earum differentialia, cuiuscunque fuerint ordinis, contineat, neque praeterea alias formulas integrales in se implicet.

Coroll. 2.

72. Quod si ergo statuamus

$$dy = p\,dx, \quad dp = q\,dx, \quad dq = r\,dx \text{ etc.}$$

vt species differentialium tollatur, quoniam integratio

tio requirit formulam differentialem expressio illa W semper reducetur ad huiusmodi formam $V dx$ existente V functione quantitatum x, y, p, q etc.

Coroll. 3.

73. Cum igitur formula integralis simplex sit huiusmodi $\int V dx$, vbi V est functio quantitatum x, y, p, q, r etc. eius indolem commodissime eius differentiale repraesentabit, si dicamus esse

$$dV = M dx + N dy + P dp + Q dq + R dr + \text{etc.}$$

Scholion.

74. Distinguo hic formulas integrales simplices a complicatis, in quibus integratio proponitur eiusmodi formularum differentialium quae iam ipsae vnam pluresue formulas integrales, inuoluunt. Veluti si littera s denotet integrale

$$\int \surd (dx^2 + dy^2) = \int dx \surd (1 + pp),$$

atque quantitas V praeter illas quantitates etiam hanc s contineat, formula integralis $\int V dx$ merito censetur complicata; cuius variatio singularia praecepta postulat deinceps exponenda. Hoc autem capite primo methodum formularum integralium simplicium variationes inueniendi tradere constitui.

Theorema 2.

75. *Variatio formulae integralis* $\int W$ *semper aequalis est integrali variationis eiusdem formulae differentialis cuius integrale proponitur; seu est*

$$\delta \int W = \int \delta W.$$

Demonstratio.

Cum variatio sit excessus, quo valor variatus cuiusque quantitatis superat eius valorem naturalem, perpendamus formulae propositae $\int W$ valorem variatum, quem obtinet, si loco variabilium x et y earundem valores suis variationibus δx et δy aucti substituantur. Cum autem tum quantitas W abeat in $W + \delta W$, formae propositae valor variatus erit

$$\int (W + \delta W) = \int W + \int \delta W$$

vnde cum sit

$$\delta \int W = \int (W + \delta W) - \int W$$

habebimus

$$\delta \int W = \int \delta W$$

vnde patet variationem integralis aequari integrali variationis.

Idem etiam hoc modo ostendi potest. Ponatur $\int W = w$, ita vt quaerenda sit variatio δw. Quia

Quia ergo sumtis differentialibus est $dw = W$, capiantur nunc variationes eritque

$$\delta dw = \delta W = d\delta w$$

ob $\delta dw = d\delta w$. Nunc vero aequatio $d\delta w = \delta W$ denuo integrata praebet

$$\delta w = \int \delta W = \delta \int W.$$

Coroll. 1.

76. Proposita ergo hac formula integrali $\int V dx$ eius variatio $\delta \int V dx$ erit

$$= \int \delta (V dx) = \int (V \delta dx + dx \delta V)$$

hincque ob $\delta dx = d\delta x$ habebitur:

$$\delta \int V dx = \int V d\delta x + \int dx \delta V.$$

Coroll. 2.

77. Posito $\delta x = \omega$ vt sit $d\delta x = d\omega$, quia est

$$\int V d\omega = V\omega - \int \omega dV,$$

in priori membro differentiale variationis dx exuitur, fietque:

$$\delta \int V dx = V \delta x - \int dV \delta x + \int dx \delta V$$

vbi prima pars ab integratione est immunis.

Scholion.

78. Quemadmodum supra ostendimus signa differentiationis d cum signo variationis δ expressioni

cuicunque praefixa inter se pro lubitu permutari posse, ita nunc videmus signum integrationis $\int$ cum signo variationis δ permutari posse, cum sit

$$\delta \int W = \int \delta W.$$

Atque hoc etiam ad integrationes repetitas patet, vt si proposita fuerit talis formula $\int\int W$ eius variatio his modis exhiberi possit

$$\delta \int\int W = \int \delta \int W = \int\int \delta W$$

ideoque variatio formularum integralium ad variationes expressionum nullam amplius integrationem inuoluentium reducatur, pro quibus inueniendis iam supra praecepta sunt tradita.

Problema 6.

79 Propositis binarum variabilium x et y variationibus δx et δy, si positis

$$dy = p\,dx,\ dp = q\,dx,\ dq = r\,dx \text{ etc.}$$

fuerit V functio quaecunque quantitatum x, y, p, q, r etc. formulae integralis $\int V dx$ variationem inuestigare.

Solutio.

Modo vidimus (77.) huius formulae integralis variationem ita exprimi vt sit

$$\delta \int V dx = V \delta x - \int dV \delta x + \int dx \delta V.$$

Iam ad variationem δV elidendam, cum sit V functio quantitatum x, y, p, q, r etc. statuamus eius

eius differentiale esse

$$dV = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr \text{ etc.}$$

ac simili modo eius variatio ita erit expressa:

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r \text{ etc.}$$

quibus valoribus substitutis consequimur variationem quaesitam:

$$\delta\int V\,dx = V\delta x + \int dx(M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r \text{ etc.})$$
$$- \int \delta x(M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr \text{ etc.})$$

vbi cum partes ab M pendentes se destruant, erit partibus secundum litteras N, P, Q, R etc. separatis variatio

$$\delta\int V\,dx = V\delta x + \int N(dx\delta y - dy\delta x) + \int P(dx\delta p - dp\delta x)$$
$$+ \int Q(dx\delta q - dq\delta x) + \int R(dx\delta r - dr\delta x)$$
$$\text{etc.}$$

vbi est vti supra inuenimus:

$$dx\delta p = d\delta y - p\,d\delta x;\quad dx\delta q = d\delta p - q\,d\delta x;$$
$$dx\delta r = d\delta q - r\,d\delta x \text{ etc.}$$

quibus valoribus substitutis ob $dy = p\,dx$ obtinetur:

$$\delta\int V\,dx = V\delta x + \int N\,dx(\delta y - p\delta x) + \int P\,d.(\delta y - p\delta x)$$
$$+ \int Q\,d.(\delta p - q\delta x) + \int R\,d.(\delta q - r\delta x)$$
$$\text{etc.}$$

Ad hanc expreſſionem vlterius reducendam, notetur eſſe

$$\delta p - q\delta x = \frac{d\delta y - p d\delta x - d p \delta x}{dx} = \frac{d.(\delta y - p\delta x)}{dx}$$
$$\delta q - r\delta x = \frac{d\delta p - q d\delta x - d q\delta x}{dx} = \frac{d.(\delta p - q\delta x)}{dx}$$
$$\delta r - s\delta x = \frac{d\delta q - r d\delta x - d r\delta x}{dx} = \frac{d.(\delta q - r\delta x)}{dx}$$

etc.

quo pacto quaeuis formula ad praecedentem reducitur; vnde ſi breuitatis gratia ponamus $\delta y - p\delta x = \omega$ erit vt ſequitur:

$$\delta y - p\delta x = \omega$$
$$\delta p - q\delta x = \frac{d\omega}{dx}$$
$$\delta q - r\delta x = \frac{1}{dx} d. \frac{d\omega}{dx}$$
$$\delta r - s\delta x = \frac{1}{dx} d. \frac{1}{dx} d. \frac{d\omega}{dx}$$

etc.

ſicque variationibus litterarum deriuatarum p, q, r etc. ex calculo excluſis adipiſcimur variationem quaeſitam

$$\delta\int V dx = V\delta x + \int N dx\omega + \int P d\omega + \int Q d. \frac{d\omega}{dx} + \int R d. \frac{1}{dx} d. \frac{d\omega}{dx}$$
$$+ \int S d. \frac{1}{dx} d. \frac{1}{dx} d. \frac{d\omega}{dx} + \int T d. \frac{1}{dx} d. \frac{1}{dx} d. \frac{1}{dx} d. \frac{d\omega}{dx}$$

etc.

cuius formae lex progreſſionis eſt manifeſta, cuiuscunque gradus differentialia in formulam V ingrediantur.

Coroll. 1.

Coroll. 1.

80. Huius igitur variationis pars prima $V\delta x$ a ſigno integrationis eſt immunis atque adeo ſolam variationem δx inuoluit, reliquae vero partes vtramque perpetuo eodem modo iunctam et in littera

$$\omega = \delta y - p\,dx$$

comprehenſam continet.

Coroll. 2.

81. Secunda pars

$$\int N\,dx.\omega = \int N\omega\,dx$$

commodius exprimi nequit, tertia vero $\int P\,d\omega$ commodius ita exprimi videtur, vt ſit

$$\int P\,d\omega = P\omega - \int \omega\,dP$$

ac poſt ſignum integrale iam ipſa quantitas ω reperiatur.

Coroll. 3.

82. Quarta pars $\int Q\,d.\frac{d\omega}{dx}$ ſimili modo reducitur ad

$$Q\frac{d\omega}{dx} - \int dQ.\frac{d\omega}{dx},$$

hocque membrum poſterius, cum ſit $\int \frac{dQ}{dx}.d\omega$ porro praebet

$$\frac{dQ}{dx}\omega - \int \omega\,d.\frac{dQ}{dx},$$

ita vt tertia pars reſoluatur in haec membra:

$$Q.\frac{d\omega}{dx} - \frac{dQ}{dx}.\omega + \int \omega\,d\frac{dQ}{dx}.$$

Coroll. 4.

Coroll. 4.

83. Quinta pars

$$\int R\, d.\frac{1}{dx}\, d.\frac{d\omega}{dx}$$

reducitur primo ad

$$R.\frac{1}{dx}d.\frac{d\omega}{dx} - \int \frac{dR}{dx} d.\frac{d\omega}{dx},$$

tum vero posterius membrum ad

$$\frac{dR}{dx}.\frac{d\omega}{dx} - \int \frac{1}{dx} d.\frac{dR}{dx}.d\omega,$$

hocque tandem vlterius ad

$$\frac{1}{dx} d.\frac{dR}{dx}.\omega - \int \omega\, d\frac{1}{dx}.d\frac{dR}{dx};$$

ita vt haec quinta pars iam ita exprimatur

$$R.\frac{1}{dx}d.\frac{d\omega}{dx} - \frac{dR}{dx}.\frac{d\omega}{dx} + \frac{1}{dx}d.\frac{dR}{dx}.\omega - \int \omega\, d.\frac{1}{dx}d.\frac{dR}{dx}.$$

Coroll. 5.

84. Simili modo sexta pars

$$\int S\, d.\frac{1}{dx}\, d.\frac{1}{dx}\, d.\frac{d\omega}{dx}$$

ita reperitur expressa:

$$S.\frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx} - \frac{dS}{dx}.\frac{1}{dx}d.\frac{d\omega}{dx} + \frac{1}{dx}d.\frac{dS}{dx}.\frac{d\omega}{dx} - \frac{1}{dx}d.\frac{1}{dx}d.\frac{dS}{dx}.\omega$$
$$+ \int \omega\, d.\frac{1}{dx}d.\frac{1}{dx}d\frac{dS}{dx}.$$

Problema

Problema 7.

85. Positis $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = sdx$ etc. si V fuerit functio quaecunque quantitatum x, y, p, q, r, s etc. ita vt sit

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc.}$$

formulae integralis $\int Vdx$ variationem ex vtriusque variabilis x et y variatione natam ita exprimere, vt post signum integrale nulla occurrant variationum differentialia.

Solutio.

In corollariis praecedentis problematis iam omnia ita sunt ad hunc scopum praeparata, vt nihil aliud opus sit, nisi vt transformationes singularum partium in ordinem redigantur, quo pacto duplicis generis partes obtinentur; vno continente formulas integrales, quas quidem omnes in eandem summam colligere licet, altero partes absolutas quas ita in membra distribuemus, vt secundum ipsas variationes δx et δy, earumque differentialia cuiusque gradus procedant. Posita autem breuitatis gratia formula $\delta y - pdx = \omega$ variatio quaesita ita se habebit:

$$\begin{aligned}
\delta\int Vdx = {} & \int \omega dx \left(N - \frac{dP}{dx} + \frac{1}{dx} d.\frac{dQ}{dx} - \frac{1}{dx} d.\frac{1}{dx} d.\frac{dR}{dx} + \frac{1}{dx} d.\frac{1}{dx} d.\frac{1}{dx} d.\frac{dS}{dx} - \text{etc.}\right) \\
& + V\delta x + \omega \left(P - \frac{dQ}{dx} + \frac{1}{dx} d.\frac{dR}{dx} - \frac{1}{dx} d \frac{1}{dx} d.\frac{dS}{dx} + \text{etc.}\right. \\
& \qquad + \frac{d\omega}{dx}\left(Q - \frac{dR}{dx} + \frac{1}{dx} d.\frac{dS}{dx} - \text{etc.}\right) \\
& \qquad + \frac{1}{dx} d.\frac{d\omega}{dx}\left(R - \frac{dS}{dx} + \text{etc.}\right) \\
& \qquad + \frac{1}{dx} d.\frac{1}{dx} d.\frac{d\omega}{dx}(S - \text{etc.})
\end{aligned}$$

cuius formae indoles ex sola inspectione statim est manifesta, vt vberiori illustratione non sit opus.

Coroll. 1.

86. Haec expressio multo simplicior redditur, si elementum dx capiatur constans, quo quidem amplitudo expressionis nequaquam restringitur, tum enim fiet:

$$\begin{aligned}\delta\int V dx = \int \omega dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right) \\ + V\delta x + \omega\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right) \\ + \frac{d\omega}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right) \\ + \frac{dd\omega}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right) \\ + \frac{d^3\omega}{dx^3}\left(S - \text{etc.}\right).\end{aligned}$$

Coroll. 2.

87. Si quaestio sit de linea curua prima pars integralis valorem per totam curuam ab initio vsque ad terminum, vbi coordinatae x et y subsistunt, congregat, simul omnes variationes in singulis curvae punctis factas complectens, dum reliquae partes absolutae tantum per variationes in extremitate curuae factas definiuntur.

Coroll. 3.

88. Si curuam coordinatis x et y definitam vt datam spectemus, aliaque curua ab ea infinite

parum

parum discrepans consideretur, dum in singulis punctis vtrique coordinatae variationes quaecunque tribuantur, expressio inuenta indicat, quantum formulae integralis $\int V dx$ valor ex curua variata collectus superat eiusdem valorem ex ipsa curua data desumtum.

Coroll. 4.

89. Cum sit $\omega = \delta y - p \delta x$, patet hanc quantitatem ω euanescere, si in singulis punctis variationes δx et δy ita accipiantur, vt sit

$$\delta y : \delta x = p : 1 = dy : dx.$$

Hoc igitur casu curua variata plane non discrepat a data, ac tota variatio formulae $\int V dx$ reducitur ad $V \delta x$.

Scholion 1.

90. Variatio haec pro formula integrali $\int V dx$ inuenta statim suppeditat regulam, quam olim tradidi pro curua inuenienda in qua valor eiusdem formulae integralis sit maximus vel minimus. Illa enim regula postulat, vt haec expressio

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}$$

nihilo aequalis statuatur. Hic autem statim euidens est, ad id, vt variatio formulae $\int V dx$ euanescat, quemadmodum natura maximorum et minimorum exigit, ante omnia requiri, vt prima pars signo integrali contenta euanescat, ex quo fit:

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0.$$

 Prae-

Praeterea vero etiam partes abſolutas nihilo aequari oportet, in quo applicatio ad vtrumque curuae terminum continetur. Ipſa enim curuae natura per illam aequationem exprimitur, quae cum ob differentialia altioris gradus totidem integrationes totidemque conſtantes arbitrarias aſſumat, harum conſtantium determinationi illae partes abſolutae inſerviunt, vt tam in initio quam in fine quaeſita curua certis conditionibus reſpondeat, veluti ad datas lineas curuas terminetur. Ac ſi aequatio illa fuerit differentialis ordinis quarti vel adeo altioris, partium quoque abſolutarum numerus augetur, quibus effici poteſt, vt curua quaeſita non ſolum vtrinque ad datas lineas terminetur, ſed ibidem quoque certa directio, quin etiam ſi ad altiora differentialia aſſurgat, certa curuaminis lex praeſcribi queat. Semper autem applicationem faciendo pulcherrime euenire ſolet, vt ipſa quaeſtionum indoles eiusmodi conditiones inuoluat, quibus per partes abſolutas commodiſſime ſatisfieri poſſit.

Scholion 2.

91. Quanta autem myſteria in hac forma, quam pro variatione formulae integralis $\int V dx$ invenimus lateant, in eius applicatione ad maxima et minima multo luculentius declarare licet, hic tantum obſeruo, partem integralem neceſſario in iſtam variationem ingredi. Cum enim rem in latiſſimo

ſenſu

fenfu fimus complexi, vt in fingulis curuae punctis vtrique variabili x et y variationes quascunque nulla plane lege inter fe connexas tribuerimus, fieri omnino nequit, vt variatio toti curuae conueniens non fimul ab omnibus variationibus intermediis pendeat, quippe quibus aliter conftitutis neceffe eft, vt inde totius curuae variatio mutationem perpetuatur. Atque in hoc variatio formularum integralium potiffimum diffi'et a variatione eiusmodi expreffionum, quales in fuperiori capite confideranimus, quae vnice a variatione vltimis elementis tributa pendet. Ex quo luculenter fequitur, fi forte quantitas V ita fuerit comparata, vt formula differentialis $V dx$ integrationem admittat, nulla ftabilita relatione inter variabiles x et y fique integralis $\int V dx$ fit functio abfoluta quantitatum x, y, p, q, r etc. tum etiam eius variationem tantum a variatione extremorum elementorum pendere poffe, ficque partem variationis integralem plane in nihilum abire debere ex quo fequens infigne Theorema colligitur.

Theorema 3.

92. *Pofito* $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = sdx$ etc. *fi* V *fuerit eiusmodi functio ipfarum* x, y, p, q, r, s etc. *vt pofito eius differentiali*

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \text{ etc.}$$

fuerit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

sumto elemento dx *constante*, *tum formula differentialis* Vdx *per se erit integrabilis*, *nulla stabilita relatione inter variabiles* x *et* y; *ac vicissim.*

Demonstratio.

Si fuerit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0,$$

tum formulae integralis $\int V dx$ variatio nullam implicat formulam integralem, ideoque pro quouis situ coordinatarum x et y a solis variationibus, quae ipsis in extremo termino tribuuntur, pendet, quod fieri neutiquam posset, si formula $V dx$ integrationem respueret, propterea quod tum variatio insuper ab omnibus variationibus intermediis simul necessario penderet; vnde sequitur quoties aequatio illa locum habet, toties formulam $V dx$ integrationem admittere; ita vt $\int V dx$ futura sit certa ac definita functio quantitatum x, y, p, q, r, s etc. Vicissim autem quoties formula differentialis $V dx$ integrationem admittit, eiusque propterea integrale $\int V dx$ est vera functio quantitatum x, y, p, q, r, s etc. toties quoque eius variatio tantum ab extremis variationibus ipsarum x et y pendet, neque variationes intermediae eam vllo modo afficere possunt. Ex quo necesse est vt variationis pars integralis supra inuenta euanescat, id quod fieri nequit, nisi fuerit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

sicque Theorema propositum etiam inuersum veritati est consentaneum.

Coroll. 1.

Coroll. 1.

93. En ergo insigne criterium, cuius ope formula differentialis duarum variabilium, cuiuscunque gradus differentialia in eam ingrediantur, diiudicari potest, vtrum sit integrabilis nec ne? Multo latius ergo patet illo criterio satis noto, quo formularum differentialium primi gradus integrabilitas dignosci solet.

Coroll. 2.

94. Primo ergo si quantitas V fit tantum functio ipsarum x et y nullam differentialium rationem inuoluens, vt sit

$$dV = M\,dx + N\,dy,$$

tum formula differentialis $V\,dx$ integrationem non admittit, nisi sit $N = 0$, hoc est nisi V sit functio ipsius x tantum, quod quidem per se est perspicuum.

Coroll. 3.

95. Proposita autem huiusmodi formula differentiali: $v\,dx + u\,dy$, ea cum forma $V\,dx$ ob $dy = p\,dx$ comparata dat $V = u + pu$; ideoque

$$M = \left(\tfrac{dv}{dx}\right) + p\left(\tfrac{du}{dx}\right);\ N = \left(\tfrac{dv}{dy}\right) + p\left(\tfrac{du}{dy}\right)$$

et $P = u$, quandoquidem quantitates v et u nulla differentialia implicare sumuntur. Erit ergo

$$dP = du = dx\left(\tfrac{du}{dx}\right) + dy\left(\tfrac{du}{dy}\right).$$

Quare

Quare cum criterium integrabilitatis postulet vt sit

$$N - \frac{dP}{dx} = 0,$$

erit pro hoc casu

$$\left(\frac{dv}{dy}\right) + p\left(\frac{dx}{dy}\right) - \left(\frac{du}{dx}\right) - p\left(\frac{du}{dy}\right) = 0$$

seu $\left(\frac{dv}{dy}\right) = \left(\frac{du}{dx}\right)$,

quod est criterium iam vulgo cognitum.

Scholion 1.

96. Demonstratio huius Theorematis omnino est singularis, cum ex doctrina variationum sit petita, quae tamen ab hoc argumento prorsus est aliena; vix vero alia via patet ad eius demonstrationem pertingendi. Tum vero hic accuratior cognitio functionum diligenter est animaduertenda, qua ostendimus formulam integralem $\int V dx$ neutiquam vt functionem quantitatum x, y, p, q, r etc. spectari posse, nisi reuera integrationem admittat. Natura enim functionum semper hanc proprietatem habet adiunctam, vt statim atque quantitatibus, quae eam ingrediuntur valores determinati tribuuntur, ipsa functio ex iis formata determinatum adipiscatur valorem; veluti haec functio xy, si ponamus $x = 2$ et $y = 3$, valorem accipit $= 6$. Longe secus autem euenit in formula integrali $\int y dx$, cuius valor pro casu $x = 2$ et $y = 3$ neutiquam assignari potest, nisi inter y et x certa quaedam relatio

tio ſtatuatur; tum autem ea formula abit in functionem vnicae variabilis. Formularum ergo integralium, quae integrari nequeunt, natura ſollicite a natura functionum diſtingui debet, cum functiones ſtatim atque quantitatibus variabilibus, ex quibus conflantur, determinati valores tribuuntur, ipſae quoque determinatos valores recipiant, etiamſi variabiles nullo modo a ſe inuicem pendeant; quod minime euenit in formulis integralibus, quippe quarum determinatio omnes plane valores intermedios ſimul includit. Imprimis autem huic diſcrimini vniuerſa doctrina de maximis et minimis, ad quam hic attendimus, innitur, vbi formulas, quibus maximi minimiue proprietas conciliari debet; neceſſario eiusmodi integrales eſſe oportet, quae per ſe integrationem non admittant.

Scholion 2.

97. Ad maiorem Theorematis illuſtrationem eiusmodi formulam integralem $\int V dx$ conſideremus, quae per ſe ſit integrabilis, ponamusque exempli gratia

$$\int V dx = \frac{x\,dy}{y\,dx} = \frac{xp}{y},$$

ita vt ſit

$$V = \frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y},$$

atque ideo haec formula differentialis

$$\left(\frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y}\right) dx$$

fit abfolute integrabilis; ac videamus, an Theorema noftrum hanc integrabilitatem declaret? Quantitatem ergo V differentiemus, et differentiali cum forma

$$dV = M\,dx + N\,dy + P\,dp + Q\,dq$$

comparato obtinebimus:

$$M = \frac{-pp}{yy} + \frac{q}{y};\; N = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy};\; P = \frac{1}{y} - \frac{2xp}{yy} \text{ et } Q = \frac{x}{y}.$$

Cum nunc fecundum Theorema fieri debeat

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0,$$

primo colligimus differentiando:

$$\frac{dP}{dx} = \frac{-2p}{yy} + \frac{4xpp}{y^3} - \frac{2xq}{yy} \text{ et } \frac{dQ}{dx} = \frac{1}{y} - \frac{xp}{yy}$$

tum vero

$$\frac{ddQ}{dx^2} = \frac{-2p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy}.$$

Ergo

$$\frac{dP}{dx} - \frac{ddQ}{dx^2} = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy}$$

cui valori quantitas N vtique eft aequalis.

Scholion 3.

98. Ceterum quando formula differentialis $V\,dx$ integrationem per fe admittit, ideoque pofito

$$dV = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + \text{etc.}$$

fecundum Theorema eft

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

VV hinc

hinc alia insignia consectaria deducuntur. Primo enim cum per dx multiplicando et integrando fiat

$$\int N\,dx - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \text{etc.} = A$$

patet etiam formulam $N\,dx$ absolute esse integrabilem. Deinde cum hinc porro fiat

$$\int dx\left(\int N\,dx - P\right) + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.} = Ax + B$$

etiam formula

$$dx\left(\int N\,dx - P\right)$$

integrationem admittit. Postea etiam simili modo integrabilis erit haec forma

$$dx\left(\int dx\left(\int N\,dx - P\right) + Q\right)$$

tum vero etiam haec

$$dx\left(\int dx\left(\int N\,dx - P\right) + Q\right) - R$$

et ita porro. Vnde sequens Theorema non minus notatu dignum et in praxi vtilissimum colligimus.

Theorema 4.

99. *Positis* $dy = p\,dx$, $dp = q\,dx$, $dq = r\,dx$, $dr = s\,dx$ etc. *si* V *eiusmodi fuerit functio ipsarum* x, y, p, q, r, s etc. *vt formula differentialis* $V\,dx$ *per se sit integrabilis, tum posito*

$$dV = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds + \text{etc.}$$

etiam sequentes formulae differentiales per se integrationem admittent:

I. Formula $N\,dx$ erit per se integrabilis
tum posito $P - \int N\,dx = \mathfrak{P}$.

II. Formula $\mathfrak{P}\,dx$ erit per se integrabilis
porro posito $Q - \int \mathfrak{P}\,dx = \mathfrak{Q}$.

III. Formula $\mathfrak{Q}\,dx$ erit per se integrabilis
deinde posito $R - \int \mathfrak{Q}\,dx = \mathfrak{R}$.

IV. Formula $\mathfrak{R}\,dx$ erit per se integrabilis
vlterius posito $S - \int \mathfrak{R}\,dx = \mathfrak{S}$.

V. Formula $\mathfrak{S}\,dx$ erit per se integrabilis
et ita porro.

Demonstratio.

Huius Theorematis veritas iam in praeced. §. est euicta, vnde simul patet, si omnes hae formulae integrationem admittant etiam principalem $V\,dx$ absolute fore integrabilem.

Coroll. 1.

100. Cum V sit functio quantitatum

$$x, y, p = \frac{dy}{dx};\ q = \frac{dp}{dx};\ r = \frac{dq}{dx} \text{ etc.}$$

quantitates per differentiationem inde deriuatae M, N, P, Q, R etc. etiam ita exhiberi possunt vt sit

$$M = \left(\frac{dV}{dx}\right);\ N = \left(\frac{dV}{dy}\right);\ P = \left(\frac{dV}{dp}\right);\ Q = \left(\frac{dV}{dq}\right) \text{ etc.}$$

vnde

vnde ob primam formulam patet, si fuerit formula $V dx$ integrabilis, tum etiam formulam $\left(\frac{dV}{dy}\right)dx$ fore integrabilem.

Coroll. 2.

101. Deinde ergo quoque ob eandem rationem formula haec: $\left(\frac{ddV}{dy^2}\right)dx$, hincque porro istae

$$\left(\frac{d^3V}{dy^3}\right)dx\,,\ \left(\frac{d^4V}{dy^4}\right)dx \text{ etc.}$$

omnes per se integrationem admittent.

Coroll. 3.

102. Quia tot tantum litterae P, Q, R etc. adsunt, quoti gradus differentialia in formula $V dx$ repetriuntur, et sequentes omnes euanescunt, litterae germanicae inde deriuatae $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{S}$ etc. tandem euanescere vel in functiones solius quantitatis x abire debent, quia alioquin sequentes integrabilitates locum habere non possent.

Exemplum.

103. *Sit* V *eiusmodi functio, vt fiat*

$$\int V dx = \frac{y(dx^2+dy^2)^{\frac{3}{2}}}{x\,dx\,ddy}.$$

Factis substitutionibus

$$dy = p\,dx\,,\ dp = q\,dx\,,\ dq = r\,dx \text{ etc.}$$

pro hoc exemplo functio V ita exprimetur:

$$V = \frac{p(1+pp)^{\frac{3}{2}}}{xq} - \frac{y(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3yp\sqrt{(1+pp)}}{x} - \frac{yr(1+pp)^{\frac{3}{2}}}{xqq}$$

vnde per differentiationem elicuimus sequentes valores:

$$N = \frac{-(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3p\sqrt{(1+pp)}}{x} - \frac{r(1+pp)^{\frac{3}{2}}}{xqq}$$

$$P = \frac{(1+4pp)\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq}$$

$$Q = \frac{-p(1+pp)^{\frac{3}{2}}}{xqq} + \frac{y(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr(1+pp)^{\frac{3}{2}}}{xq^3}$$

$$R = \frac{-y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Iam igitur primo integrabilem esse oportet formulam $N\,dx$ seu

$$-\frac{dx(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3p\,dx\sqrt{(1+pp)}}{x} - \frac{dq(1+pp)^{\frac{3}{2}}}{xqq}$$

vnde statim patet integrale hoc fore:

$$\int N\,dx = \frac{(1+pp)^{\frac{3}{2}}}{xq}.$$

Iam pro secunda formula hinc nanciscimur:

$$\mathfrak{P} = P - \int N dx = \frac{3pp\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq}$$

ita vt integranda sit haec formula:

$$\mathfrak{P} dx = \frac{3pdy\sqrt{(1+pp)}}{xq} - \frac{3ypdx\sqrt{(1+pp)}}{xxq} + \frac{3ydx(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypdq\sqrt{(1+pp)}}{xqq}$$

cuius integrale, vel saltem eius pars ex postremo membro manifesto colligitur: $\frac{3yp\sqrt{(1+pp)}}{xq}$, cuius differentiale cum totam formulam exhauriat erit

$$\int \mathfrak{P} dx = \frac{3yp\sqrt{(1+pp)}}{qx}.$$

Nunc pro tertia formula habebimus

$$\mathfrak{Q} = Q - \int \mathfrak{P} dx = \frac{-p(1+pp)^{\frac{3}{2}}}{xqq} + \frac{y(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr(1+pp)^{\frac{3}{2}}}{xq^{3}} - \frac{3yp\sqrt{(1+pp)}}{xq}$$

vnde per dx multiplicando ob $dx = \frac{dp}{q}$ in vltimo membro fit

$$\mathfrak{Q} dx = \frac{-dy(1+pp)^{\frac{3}{2}}}{xqq} + \frac{ydx(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2ydq(1+pp)^{\frac{3}{2}}}{xq^{3}} - \frac{3ypdp\sqrt{(1+pp)}}{xqq}$$

cuius

cuius penultimum membrum declarat integrale

$$\int \Omega\, dx = \frac{-y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Quarta porro formula ita erit comparata:

$$\mathfrak{R} = R - \int \Omega\, dx = 0,$$

vnde perspicuum est non solum hanc $\mathfrak{R}\, dx$ sed etiam sequentes omnes fore integrabiles.

Scholion.

104. Theoremata haec eo pulcriora videntur, quod eorum demonstratio eiusmodi principio innititur, cuius ratio hinc prorsus est aliena; propterea quod in his veritatibus nullum amplius vestigium variationum apparet; ex quo nullum est dubium quin demonstratio etiam ex alio fonte magis naturali hauriri queat.

CAPVT IV.

DE VARIATIONE FORMVLARVM INTEGRALIVM COMPLICATARVM DVAS VARIABILES INVOLVENTIVM.

Problema 8.

105.

Posito $v = \int \mathfrak{V} dx$; existente $\mathfrak{V}$ functione quacunque binarum variabilium x, y earumque differentialium

$$dy = p dx, \; dp = q dx, \; dq = r dx \text{ etc.}$$

si V denotet functionem quamcunque ipsius v, investigare variationem formulae integralis complicatae $\int V dx$.

Solutio.

Quia quantitas v ipsa est formula integralis $\int \mathfrak{V} dx$, formula $\int V dx$ est vtique complicata. Cum igitur functio V solam quantitatem v inuoluere ponatur, statuamus $dV = L dv$, tum vero pro functione $\mathfrak{V}$ sit eius differentiale

$$d\mathfrak{V} = \mathfrak{M} dx + \mathfrak{N} dy + \mathfrak{P} dp + \mathfrak{Q} dq + \mathfrak{R} dr + \text{ etc.}$$

 His

His positis cum variatio quaesita sit

$$\delta\int V dx = \int\delta (V dx) = \int(\delta V dx + V d\delta x),$$

et per reductionem supra adhibitam:

$$\delta\int V dx = V\delta x + \int(dx\delta V - dV\delta x).$$

Cum autem per hypothesin sit $dV = L dv$ erit etiam pro variatione $\delta V = L\delta v$, verum ob $v = \int\mathfrak{V}dx$ erit primo $dv = \mathfrak{V}dx$ ideoque $dV = L\mathfrak{V}dx$, tum vero

$$\delta v = \delta\int\mathfrak{V}dx = \mathfrak{V}\delta x + \int(dx\delta\mathfrak{V} - d\mathfrak{V}\delta x),$$

ac propterea

$$\delta V = L\mathfrak{V}\delta x + L\int(dx\delta\mathfrak{V} - d\mathfrak{V}\delta x),$$

hincque

$$\delta\int V dx = V\delta x + \int(L\mathfrak{V}dx\delta x + L dx\int(dx\delta\mathfrak{V} - d\mathfrak{V}\delta x) - L\mathfrak{V}dx\delta x)$$

seu $\delta\int V dx = V\delta x + \int L dx\int(dx\delta\mathfrak{V} - d\mathfrak{V}\delta x)$.

Ex praecedente autem capite patet esse

$$\begin{aligned}\int(dx\delta\mathfrak{V} - d\mathfrak{V}\delta x) = \delta\int\mathfrak{V}dx - \mathfrak{V}\delta x = \int\omega dx\Big(\mathfrak{N} &- \tfrac{d\mathfrak{P}}{dx} + \tfrac{dd\mathfrak{Q}}{dx^2} - \tfrac{d^3\mathfrak{R}}{dx^3} + \tfrac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\Big)\\ &+ \omega\Big(\mathfrak{P} - \tfrac{d\mathfrak{Q}}{dx} + \tfrac{dd\mathfrak{R}}{dx^2} - \tfrac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\Big)\\ &+ \tfrac{d\omega}{dx}\Big(\mathfrak{Q} - \tfrac{d\mathfrak{R}}{dx} + \tfrac{dd\mathfrak{S}}{dx^2} - \text{etc.}\Big)\\ &+ \tfrac{dd\omega}{dx^2}\Big(\mathfrak{R} - \tfrac{d\mathfrak{S}}{dx} + \text{etc.}\Big)\\ &\text{etc.}\end{aligned}$$

sumto elemento dx constante et posito breuitatis ergo $\omega = \delta y - p\delta x$. Verum cum hinc substitutio molestias pariat praestabit ex primo fonte rem repetere; cum igitur ex differentiali et variatione quantitatis $\mathfrak{V}$ fiat

$$dx\delta\mathfrak{V}-d\mathfrak{V}\delta x=dx(\mathfrak{M}\delta x+\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.})$$
$$-\delta x(\mathfrak{M}dx+\mathfrak{N}dy+\mathfrak{P}dp+\mathfrak{Q}dq+\mathfrak{R}dr+\text{etc.})$$

ob $dy=pdx$, $dp=qdx$, $dq=rdx$, $dr=sdx$ etc. erit

$$dx\delta\mathfrak{V}-d\mathfrak{V}\delta x=\mathfrak{N}dx(\delta y-p\delta x)+\mathfrak{P}dx(\delta p-q\delta x)$$
$$+\mathfrak{Q}dx(\delta q-r\delta x)+\text{etc.}$$

Verum ob dx conſtans ex §. 79. fit:

$$\delta y-p\delta x=\omega;\ \delta p-q\delta x=\frac{d\omega}{dx};\ \delta q-r\delta x=\frac{dd\omega}{dx^2};$$
$$\delta r-s\delta x=\frac{d^3\omega}{dx^3}\ \text{etc.}$$

ſicque habebitur:

$$dx\delta\mathfrak{V}-d\mathfrak{V}\delta x=\mathfrak{N}\omega dx+\mathfrak{P}d\omega+\mathfrak{Q}\frac{dd\omega}{dx}+\mathfrak{R}\frac{d^3\omega}{dx^2}$$
$$\mathfrak{S}\frac{d^4\omega}{dx^3}+\text{etc.}$$

cuius quidem integrale praebet ſuperiorem expreſſionem. Ponatur nunc integrale $\int Ldx=I$, eritque

$$\delta\int Vdx=V\delta x+I\int(dx\delta\mathfrak{V}-d\mathfrak{V}\delta x)-\int I(dx\delta\mathfrak{V}-d\mathfrak{V}\delta x).$$

Nunc vero facile colligitur fore

$$\int I(dx\delta\mathfrak{V}-d\mathfrak{V}\delta x)=\int\omega dx\left(I\mathfrak{N}-\frac{d.I\mathfrak{P}}{dx}+\frac{dd.I\mathfrak{Q}}{dx^2}-\frac{d^3.I\mathfrak{R}}{dx^3}\ \text{etc.}\right)$$
$$+\omega\left(I\mathfrak{P}-\frac{d.I\mathfrak{Q}}{dx}+\frac{dd.I\mathfrak{R}}{dx^2}\ \text{etc.}\right)$$
$$+\frac{d\omega}{dx}\left(I\mathfrak{Q}-\frac{d.I\mathfrak{R}}{dx}\ \text{etc.}\right)$$

 vnde

vnde facta substitutione concluditur variatio quaesita:

$$\begin{aligned}
\delta\int V dx = V\delta x &+ I\int\omega dx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \text{etc.}\right) \\
&- \int\omega dx\left(I\mathfrak{N} - \frac{d.I\mathfrak{P}}{dx} + \frac{dd.I\mathfrak{Q}}{dx^2} - \frac{d^3.I\mathfrak{R}}{dx^3} + \text{etc.}\right) \\
&+ I\omega\left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \frac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right) \\
&- \omega\left(I\mathfrak{P} - \frac{d.I\mathfrak{Q}}{dx} + \frac{dd.I\mathfrak{R}}{dx^2} - \frac{d^3.I\mathfrak{S}}{dx^3} + \text{etc.}\right) \\
&+ \frac{I d\omega}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \frac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right) \\
&- \frac{d\omega}{dx}\left(I\mathfrak{Q} - \frac{d.I\mathfrak{R}}{dx} + \frac{dd.I\mathfrak{S}}{dx^2} - \text{etc.}\right) \\
&+ \frac{I dd\omega}{dx^2}\left(\mathfrak{R} - \frac{d\mathfrak{S}}{dx} + \text{etc.}\right) \\
&- \frac{dd\omega}{dx^2}\left(I\mathfrak{R} - \frac{d.I\mathfrak{S}}{dx} + \text{etc.}\right) \\
&+ \frac{I d^3\omega}{dx^3}\left(\mathfrak{S} - \text{etc.}\right) \\
&- \frac{d^3\omega}{dx^3}\left(I\mathfrak{S} - \text{etc.}\right)
\end{aligned}$$

Si hic partes binae priores differentiatae iterum integrentur reliquarum facta reductione impetrabimus loco dI valorem Ldx restituendo

$$\begin{aligned}
\delta\int V dx = V\delta x &+ \int L dx\int\omega dx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \text{etc.}\right) \\
&+ \int\omega dx\left(L\mathfrak{P} - \frac{Ld\mathfrak{Q} - d.I\mathfrak{Q}}{dx} + \frac{Ldd\mathfrak{R} + d.Ld\mathfrak{R} + dd.I\mathfrak{R}}{dx^2} - \text{etc.}\right) \\
&+ \omega\left(L\mathfrak{Q} - \frac{Ld\mathfrak{R} - d.I\mathfrak{R}}{dx} + \frac{Ldd\mathfrak{S} + d.Ld\mathfrak{S} + dd.I\mathfrak{S}}{dx^2} - \text{etc.}\right) \\
&+ \frac{d\omega}{dx}\left(L\mathfrak{R} - \frac{Ld\mathfrak{S} - d.I\mathfrak{S}}{dx} + \text{etc.}\right) \\
&+ \frac{dd\omega}{dx^2}\left(L\mathfrak{S} - \text{etc.}\right)
\end{aligned}$$

quae

quae forma videtur ſimpliciſſima et ad vſum maxime accommodata.

Coroll. 1.

106. Si eiusmodi relatio inter x et y quaeratur, vt integrale $\int V dx$ maximum minimumue euadat, variationis partes integrales nihilo aequari oportet, quod in genere fieri nequit, ſed ad terminum, quousque integrale $\int V dx$ extenditur, ſpectari oportet, pro quo ſi ponamus fieri $I = \int L dx = A$, ex priori forma colligimus hanc aequationem:

$$0 = (A - I)\mathfrak{N} - \frac{d.(A-I)\mathfrak{P}}{dx} + \frac{dd.(A-I)\mathfrak{Q}}{dx^2} - \frac{d^3.(A-I)\mathfrak{R}}{dx^3} + \text{etc.}$$

Coroll. 2.

107. Quomodocunque autem haec aequatio pro quouis caſu oblato tractetur, ſemper tandem eo eſt deueniendum vt formula integralis $I = \int L dx$ per differentiationem exturbari debeat, qua operatione ſimul quantitatem A inde extrudi euidens eſt; ſicque aequatio reſultans non amplius a termino integrationis pendebit.

Coroll. 3.

108. Quod ſi in genere pro variatione formulae integralis $\int V dx$ inuenienda, valorem $\int L dx = I$ toti integrali reſpondentem ponamus $= A$ variatio

quaesita ita exprimetur:

$$\delta\int V dx = V\delta x + \int\omega dx\Big((A-I)\mathfrak{N} - \frac{d.(A-I)\mathfrak{P}}{dx} + \frac{dd.(A-I)\mathfrak{Q}}{dx^2} - \frac{d^3.(A-I)\mathfrak{R}}{dx^3} + \text{etc.}\Big)$$
$$+\omega\Big(L\mathfrak{Q} - \frac{Ld\mathfrak{R} - d.L\mathfrak{R}}{dx} + \frac{Ldd\mathfrak{S} + d.Ld\mathfrak{S} + dd.L\mathfrak{S}}{dx^2} - \text{etc.}\Big)$$
$$+\frac{d\omega}{dx}\Big(L\mathfrak{R} - \frac{Ld\mathfrak{S} - d.L\mathfrak{S}}{dx} + \text{etc.}\Big)$$
$$+\frac{dd\omega}{dx^2}\big(L\mathfrak{S} - \text{etc.}\big)$$

vbi $A-I$ est valor formulae $\int L dx$ a termino integrationis extremo ad quemuis locum indefinitum medium retro sumtus.

Scholion.

109. In solutione huius problematis compendium se obtulit, quo etiam analysis in superiori capite adhibita non mediocriter contrahi potest. Cum enim ibi (79.) peruenissemus ad

$$\delta\int V dx = V\delta x + \int(dx\delta V - dV\delta x) \text{ ob}$$

$$dV = M dx + N dy + P dp + Q dq + R dr + \text{etc. et}$$

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}$$

erit

$$dV = dx(M + Np + Pq + Qr + Rs + \text{etc.})$$

hincque colligitur

$$dx\delta V - dV\delta x = dx(N(\delta y - p\delta x) + (P\delta p - q\delta x)$$
$$+ Q(\delta q - r\delta x) + \text{etc.})$$

Iam

Iam ſi breuitatis gratia ponatur $\delta y - p\delta x = \omega$ erit differentiando

$$\delta(pdx) - qdx\delta x - p\delta dx = d\omega \text{ ; at}$$

$$\delta(pdx) = pd\delta x + \delta pdx \text{ , ergo}$$

$$\delta p - q\delta x = \frac{d\omega}{dx}.$$

Simili modo hanc formulam differentiando ob

$$dp = qdx \text{ et } dq = rdx \text{ fit}$$

$$qd\delta x + \delta qdx - qd\delta x - dq\delta x = dx(\delta q - r\delta x) = d.\frac{d\omega}{dx},$$

vnde perſpicuum eſt

poſito $\delta y - p\delta x = \omega$

fore $\delta p - q\delta x = \frac{d\omega}{dx}$

$\delta q - r\delta x = \frac{1}{dx}d.\frac{d\omega}{dx} = \frac{dd\omega}{dx^2}$ ſumto dx conſtante

$\delta r - s\delta x = \frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx} = \frac{d^3\omega}{dx^3}$

etc.

Quocirca erit

$$dx\delta V - dV\delta x = dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \frac{Sd^4\omega}{dx^4} + \text{etc.}\right)$$

ſiquidem differentiale dx conſtans accipiatur.

Problema 9.

110. Si fuerit $v = \int \mathfrak{V}dx$ exiſtente

$$d\mathfrak{V} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

tum vero ſit V functio quaecunque non ſolum

qnanquantitates

$$x\,,\,y\,,\,p=\frac{dy}{dx};\;q=\frac{dp}{dx};\;r=\frac{dq}{dx}\text{ etc.}$$

sed etiam ipsam formulam integralem $v=\int \mathfrak{V}dx$ implicans inuestigare variationem formulae integralis complicatae $\int V\,dx$.

Solutio.

Quoniam V est functio quantitatum v, x, y, p, q, r etc. sumatur eius differentiale quod sit

$$dV=L\,dv+M\,dx+N\,dy+P\,dp+Q\,dq+R\,dr+\text{etc.}$$

ac habebitur variatio ipsius V ita expressa

$$\delta V=L\,\delta v+M\,\delta x+N\,\delta y+P\,\delta p+Q\,\delta q+R\,\delta r+\text{etc.}$$

tum vero notetur, ob

$$dv=\mathfrak{V}dx\,,\;dy=p\,dx\,,\;dp=q\,dx\text{ etc. esse}$$

$$dV=dx(L\mathfrak{V}+M+Np+Pq+Qr+Rs+\text{etc.})$$

$$\text{et }\delta\mathfrak{V}=\mathfrak{M}\,\delta x+\mathfrak{N}\,\delta y+\mathfrak{P}\,\delta p+\mathfrak{Q}\,\delta q+\mathfrak{R}\,\delta r+\text{etc.}$$

Praeterea habemus:

$$\delta v=\int(\mathfrak{V}d\delta x+dx\,\delta\mathfrak{V})=\mathfrak{V}\delta x+\int(dx\,\delta\mathfrak{V}-d\mathfrak{V}\,\delta x)$$

vnde posito $\delta y-p\,\delta x=\omega$ erit per ante inuenta:

$$\delta v=\mathfrak{V}\delta x+\int dx(\mathfrak{N}\omega+\frac{\mathfrak{P}\,d\omega}{dx}+\frac{\mathfrak{Q}\,dd\omega}{dx^2}+\frac{\mathfrak{R}\,d^3\omega}{dx^3}+\text{etc.}).$$

vbi commoditatis ergo sumsimus dx constans.

His praeparatis cum variatio quaesita sit:

$$\delta\int V\,dx=V\,\delta x+\int(dx\,\delta V-dV\,\delta x)$$

vt

vt reductione fupra inuenta vti poffimus, ponamus:

$$dV = L\,dv + dW$$

vt fit

$$\delta V = L\,\delta v + \delta W \text{ et}$$

$$dW = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + \text{etc.}$$

Quocirca nancifcemur hanc formam:

$$\delta \int V\,dx = V\,\delta x + \int (L\,dx\,\delta v - L\,dv\,\delta x) + \int (dx\,\delta W - dW\,\delta x)$$

vbi eft

$$dx\,\delta W - dW\,\delta x = dx\left(N\omega + \frac{P\,d\omega}{dx} + \frac{Q\,dd\omega}{dx^2} + \frac{R\,d^3\omega}{dx^3} + \text{etc.}\right)$$

Tum vero eft

$$dx\,\delta v - dv\,\delta x = dx\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}\,d\omega}{dx} + \frac{\mathfrak{Q}\,dd\omega}{dx^2} + \frac{\mathfrak{R}\,d^3\omega}{dx^3} + \text{etc.}\right)$$

ob $dv\,\delta x = \mathfrak{V}\,dx\,\delta x$. Quibus fubftitutis colligitur variatio quaefita:

$$\delta \int V\,dx = V\,\delta x + \int L\,dx \int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}\,d\omega}{dx} + \frac{\mathfrak{Q}\,dd\omega}{dx^2} + \frac{\mathfrak{R}\,d^3\omega}{dx^3} + \text{etc.}\right)$$
$$+ \int dx\left(N + \frac{P\,d\omega}{dx} + \frac{Q\,dd\omega}{dx^2} + \frac{R\,d^3\omega}{dx^3} + \text{etc.}\right)$$

Quo iam hanc formam vlterius reducamus ponamus integrale $\int L\,dx = I$ ita fumtum, vt pro initio, vnde integrale $\int V\,dx$ capitur, euanefcat, pro fine autem vbi integrale $\int V\,dx$ terminatur fiat $I = A$, ficque fiet:

$$\delta \int V\,dx = V\,\delta x + A\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}\,d\omega}{dx} + \frac{\mathfrak{Q}\,dd\omega}{dx^2} + \frac{\mathfrak{R}\,d^3\omega}{dx^3} + \text{etc.}\right)$$
$$- \int I\,dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}\,d\omega}{dx} + \frac{\mathfrak{Q}\,dd\omega}{dx^2} + \frac{\mathfrak{R}\,d^3\omega}{dx^3} + \text{etc.}\right)$$
$$+ \int dx\left(N\omega + \frac{P\,d\omega}{dx} + \frac{Q\,dd\omega}{dx^2} + \frac{R\,d^3\omega}{dx^3} + \text{etc.}\right)$$

ad quam formam contrahendam ſtatuamus:

$$N + (A - I)\mathfrak{N} = N'$$
$$P + (A - I)\mathfrak{P} = P'$$
$$Q + (A - I)\mathfrak{Q} = Q'$$
$$R + (A - I)\mathfrak{R} = R'$$
$$\text{etc.}$$

vt prodeat forma illi, quam ſupra tractauimus ſimilis

$$\delta \int V\,dx = V\,\delta x + \int dx\left(N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.}\right)$$

vbi ergo ſi poſt ſignum integrale differentialia ipſius ω eliminentur, perueniemus ſecundum §. 86. ad hanc expreſſionem

$$\begin{aligned}\delta \int V\,dx = \int \omega\,dx&\left(N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + \frac{d^4S'}{dx^4} - \text{etc.}\right)\\ &+ V\,\delta x + \omega\left(P' - \frac{dQ'}{dx} + \frac{ddR'}{dx^2} - \frac{d^3S'}{dx^3} + \text{etc.}\right)\\ &+ \text{Conſt.} + \frac{d\omega}{dx}\left(Q' - \frac{dR'}{dx} + \frac{ddS'}{dx^2} - \text{etc.}\right)\\ &\qquad + \frac{dd\omega}{dx^2}\left(R' - \frac{dS'}{dx} + \text{etc.}\right)\\ &\qquad + \frac{d^3\omega}{dx^3}\left(S' - \text{etc.}\right)\end{aligned}$$

Conſtanti autem per integrationem inuectae eiusmodi valor tribui debet, vt pro initio integrationis formulae $\int V\,dx$ partes abſolutae ad nihilum redigantur, ſiquidem prima pars integralis ita ſumatur, vt pro eodem initio euaneſcat; tum vero vniuerſam expreſſionem ad finem integrationis, produci oportet pro quo iam poſuimus fieri $\int L\,dx = I = A$.

Coroll. 1.

Coroll. 1.

111. In parte integrali variabilitas per totam integrationis extenſionem debet comprehendi in partibus autem abſolutis ſufficit reſpexiſſe ad initium ac finem integrationis, pro vtroque autem termino conditiones variationis praeſcriptae ſuppeditant valores dx, ω, $\frac{d\omega}{dx}$, $\frac{dd\omega}{dx^2}$ etc. Ac poſtquam ex conditionibus initii conſtans rite fuerit determinata tum ſupereſt, vt ſingula membra ad finem integrationis accommodentur.

Coroll. 2.

112. Pro initio igitur integrationis vbi $I=0$, erit primo:

$$N'=N+A\mathfrak{N};\ P'=P+A\mathfrak{P};\ Q'=Q+A\mathfrak{Q};$$
$$R'=R+A\mathfrak{R}\ \text{etc.}$$

pro differentialibus vero ob $dI=Ldx$ erit

$$\frac{dN'}{dx}=\frac{dN}{dx}+\frac{Ad\mathfrak{N}}{dx}-L\mathfrak{N}$$

et ita de reliquis ſimilique modo pro differentialibus ſecundis

$$\frac{ddN'}{dx^2}=\frac{ddN}{dx^2}+\frac{Add\mathfrak{N}}{dx^2}-\frac{2Ld\mathfrak{N}}{dx}-\frac{\mathfrak{N}dL}{dx}.$$

Coroll. 3.

113. Pro fine autem integrationis, vbi $I=A$ fit

$$N'=N;\ P'=P;\ Q'=Q;\ R'=R\ \text{etc.}$$

 valo-

valores vero differentiales ita se habebunt:

$\frac{dN'}{dx} = \frac{dN}{dx} - L\mathfrak{N}$; $\frac{dP'}{dx} = \frac{dP}{dx} - L\mathfrak{P}$; $\frac{dQ'}{dx} = \frac{dQ}{dx} - L\mathfrak{Q}$ etc.

secundi vero gradus hoc modo:

$$\frac{ddN'}{dx^2} = \frac{ddN}{dx^2} - \frac{2Ld\mathfrak{N}}{dx} - \frac{\mathfrak{N}dL}{dx}$$

$$\frac{ddP'}{dx^2} = \frac{ddP}{dx^2} - \frac{2Ld\mathfrak{P}}{dx} - \frac{\mathfrak{P}dL}{dx}$$

et ita porro.

Scholion 1.

114. Quanquam natura variationum atque etiam quaestionum eo pertinentium iam satis est explicata, tamen huius argumenti tam dignitas quam nouitas ampliorem illustrationem requirere videntur, cum ne superfluum quidem foret eadem saepius inculcari. Cum igitur ante geometria et huius calculi applicatione ad maxima et minima vsi simus ad hanc doctrinam magis explanandam, hic rem generalius pro sola Analysi contemplabimur. Primo igitur spectatur relatio quaecunque inter binas variabiles x et y, siue ea sit cognita, siue demum definienda, indeque formata consideratur formula integralis quaecunque $\int V dx$, quae intra certos terminos comprehensa, seu integratione a dato initio ad datum finem extensa, vtique certum quendam valorem recipere debet. Tum illa relatio inter x et y, quaecunque fuerit, quomodocunque infinite parum immutetur, vt singulis x variationibus quibuscunque δx auctis iam respondeant eaedem y variationibus quoque

que quibuscunque δy auctae, vbi quidem obseruandum est tam in initio quam fine rationem harum variationum per conditiones quaestionum dari, in medio autem istas variationes ita generaliter assumi, vt nulla plane lege inter se connectantur. Tum ex hac relatione variata eiusdem formulae integralis $\int V dx$ ab eodem initio ad eundem finem expansus, seu intra eosdem terminos contentus definiri concipitur, ac tota iam quaestio in hoc versatur vt huius postremi valoris variati excessus supra priorem illum valorem formulae $\int V dx$ inuestigetur. Qui excessus cum per $\delta \int V dx$, qnae forma ipsa est variatio formulae $\int V dx$ indicetur, huius quaestionis solutionem hactenus dedimus ita late patentem, vt omnes casus quibus quantitas V est functio quaecunque non solum ipsarum x, y, p, q, r, s etc. sed etiam insuper formulam quandam integralem $v = \int \mathfrak{V} dx$ vtcunque inuoluens, in se complectatur.

Scholion 2.

115. Quod in praecedente capite tacite assumsimus de quantitate constante variationi inuentae adiicienda quippe quam pars integralis variationis sponte inuoluit, hoc in istius problematis solutione accuratius exponere est visum. Cum scilicet in huiusmodi quaestionibus, quae ad formulas integrales reducuntur, perpetuo ad terminos integrationis sit respiciendum, siquidem integrale nihil aliud est

 nisi

nisi summa elementorum a termino dato seu initio ad alium terminum seu finem continuatorum, haec consideratio prorsus essentialis est omni integrationi, sine qua idea valoris integralis ne consistere quidem potest. Quamobrem constitutis integrationis terminis initio scilicet et fine, statim ac variationis pars integralis ita est accepta vt pro initio euadat nulla, tum eiusmodi constantem adiici oportet, vt etiam partes absolutae pro eodem initio destruantur, sicque vniuersa variationis expressio ad nihilum redigatur. Quod cum fuerit factum, ad finem integrationis demum progredi licet, vt hoc pacto vera variatio formulae integralis propositae ab initio ad finem extensae obtineatur. Haec autem variationum doctrina ad duplicis generis quaestiones accommodari potest; dum in altero relatio inter variabiles x et y data assumitur, et formulae integralis itidem datae $\int V dx$ variatio inuestigatur postquam per totam integrationis extensionem variabilibus x et y variationes quaecunque fuerint tributae, in altero autem genere ipsa illa variabilium x et y relatio quaeritur, vt formulae integralis $\int V dx$ variatio certa proprietate sit praedita; quemadmodum si ea formula maximum minimumue valorem recipere debeat hanc variationem in nihilum abire necesse est. Vbi iterum duo casus se offerunt, prout maximum minimumue locum habere debet, vel quaecunque variationes ipsis x et y tribuantur, vel si tantum hae variationes certae cuidam legi adstringantur.

tur. Ex quo manifeſtum eſt hanc Theoriam multo latius patere, quam quidem ea adhuc in vſum eſt vocata.

Problema 10.

116. Si functio V praeter binas variabiles x, y cum ſuis valoribus differentialibus

$$p = \frac{dy}{dx},\ q = \frac{dp}{dx},\ r = \frac{dq}{dx} \text{ etc.}$$

etiam duas pluresue formulas integrales

$$v = \int \mathfrak{V} dx;\ v' = \int \mathfrak{V}' dx \text{ etc.}$$

inuoluat vt ſit

$$d\mathfrak{V} = \mathfrak{M}\,dx + \mathfrak{N}\,dy + \mathfrak{P}\,dp + \mathfrak{Q}\,dq + \mathfrak{R}\,dr \text{ etc.}$$

$$d\mathfrak{V}' = \mathfrak{M}'dx + \mathfrak{N}'dy + \mathfrak{P}'dp + \mathfrak{Q}'dq + \mathfrak{R}'dr \text{ etc.}$$

atque differentiali ſumto

$$dV = L\,dv + L'dv' + M\,dx + N\,dy + P\,dp + Q\,dq + \text{etc.}$$

inuenire variationem formulae integralis $\int V dx$.

Solutio.

Si huius problematis ſolutio eodem modo inſtituatur, ac praecedentis, mox patebit calculum a geminata formula integrali

$$v = \int \mathfrak{V} dx \quad \text{et} \quad v' = \int \mathfrak{V}' dx$$

non turbari neque etiam ſi plures eiusmodi inuoluerentur. Quare tota ſolutio tandem huc redibit, vt conſtitutis integrationis terminis primo integralia

$$\int L\,dx = I \quad \text{et} \quad \int L'dx = I'$$

ita

ita fint capienda, vt pro initio integrationis euanefcant, tum vero pro fine integrationis fiat $I = A$ et $I' = A'$; quibus quantitatibus inuentis ftatuatur porro:

$$N + (A - I)\mathfrak{N} + (A' - I')\mathfrak{N}' = N'$$
$$P + (A - I)\mathfrak{P} + (A' - I')\mathfrak{P}' = P'$$
$$Q + (A - I)\mathfrak{Q} + (A' - I')\mathfrak{Q}' = Q'$$
$$R + (A - I)\mathfrak{R} + (A' - I')\mathfrak{R}' = R'$$
etc.

eritque variatio quaefita, dum vtrique variabili x et y variationes quaecunque tribuuntur, ex praeced. Solution.

$$\begin{aligned}\delta\int V\,dx = \int \omega\,dx\Big(N' - \tfrac{dP'}{dx} + \tfrac{ddQ'}{dx^2} - \tfrac{d^3R'}{dx^3} + \tfrac{d^4S'}{dx^4} - \text{etc.}\Big)\\ + V\delta x + \omega\Big(P' - \tfrac{dQ'}{dx} + \tfrac{ddR'}{dx^2} - \tfrac{d^3S'}{dx^3} + \text{etc.}\Big)\\ + \text{Conft.} + \tfrac{d\omega}{dx}\Big(Q' - \tfrac{dR'}{dx} + \tfrac{ddS'}{dx^2} - \text{etc.}\Big)\\ + \tfrac{dd\omega}{dx^2}\Big(R' - \tfrac{dS'}{dx} + \text{etc.}\Big)\\ + \tfrac{d^3\omega}{dx^3}\big(S' - \text{etc.}\big)\end{aligned}$$

vbi commoditatis gratia elementum dx conftans eft affumtum.

Corollarium.

117. Si ergo etiam plures huiusmodi formulae integrales $\int \mathfrak{V}\,dx$ in functionem V quomodocunque ingrediantur; expreffio variationis quaefitae inde non mutatur, fed tantum quantitates N', P', Q', R', etc. ex iis rite definiri conuenit.

Scho-

Scholion.

118. Etsi formulae integrales

$$I = \int L dx, \quad I' = \int L' dx$$

binas variabiles inuoluunt, ideoque valores fixos recipere non posse videntur, tamen perpendendum est, in omnibus huiusmodi quaestionibus semper certam quandam relationem inter binas variabiles x et y supponi, siue ea absolute detur, siue demum per calculum definiri debeat. Hac igitur ipsa relatione iam in vsum vocata, vt quantitas y instar functionis ipsius x spectari possit, formulae illae integrales vtique determinatos valores sortientur.

Problema 11.

119. Si functio $\mathfrak{V}$ praeter variabiles x et y, earumque valores differentiales p, q, r, s etc. ipsam quoque formulam integralem $u = \int \mathfrak{v} dx$ inuoluat, vt eius differentiale sit

$$d\mathfrak{V} = \mathfrak{L} du + \mathfrak{M} dx + \mathfrak{N} dy + \mathfrak{P} dp + \mathfrak{Q} dq + \mathfrak{R} dr + \text{etc.}$$

existente

$$d\mathfrak{v} = \mathfrak{m} dx + \mathfrak{n} dy + \mathfrak{p} dp + \mathfrak{q} dq + \mathfrak{r} dr + \text{etc.}$$

tum vero sit V functio quaecunque ipsarum x, y, p, q, r etc. insuperque formulae integralis $v = \int \mathfrak{V} dx$, vt sit

$$dV = L dv + M dx + N dy + P dp + Q dq + R dr \text{ etc.}$$

inuenire variationem formulae integralis $\int V dx$.

Solutio.

Ex problemate 9. ſtatim inuenimus variationem formulae integralis $\int \mathfrak{V} dx = v$; conſtitutis enim integrationis terminis ſumtoque integrali $\int \mathfrak{L} dx = \mathfrak{I}$ ita vt euaneſcat pro integrationis initio, pro fine fiat $\mathfrak{I} = \mathfrak{A}$, tum fiat breuitatis gratia:

$$\mathfrak{N} + (\mathfrak{A} - \mathfrak{I})\mathfrak{n} = \mathfrak{N}'; \ \mathfrak{P} + (\mathfrak{A} - \mathfrak{I})\mathfrak{p} = \mathfrak{P}';$$
$$\mathfrak{Q} + (\mathfrak{A} - \mathfrak{I})\mathfrak{q} = \mathfrak{Q}' \text{ etc.}$$

erit ex illius problematis ſolutione:

$$\delta v = \mathfrak{V}\delta x + \int dx\left(\mathfrak{N}'\omega + \frac{\mathfrak{P}' d\omega}{dx} + \frac{\mathfrak{Q}' dd\omega}{dx^2} + \frac{\mathfrak{R}' d^3\omega}{dx^3} + \text{etc.}\right)$$

poſito $\omega = \delta y - p\delta x$ et ſumto dx conſtante.

Iam vero cum quaeratur $\delta \int V dx$ ob

$$\delta \int V dx = V\delta x + \int (dx\delta V - dV\delta x)$$

poſito breuitatis ergo:

$$dV = L dv + dW \text{ et } \delta V = L\delta v + \delta W$$

vt ſit

$$dW = M dx + N dy + P dp + Q dq + R dr + \text{etc.}$$

erit vt ibidem vidimus:

$$\delta \int V dx = V\delta x + \int (L dx\delta v - L dv\delta x)$$
$$+ \int dx\left(N\omega + \frac{P d\omega}{dx} + \frac{Q dd\omega}{dx^2} + \frac{R d^3\omega}{dx^3} + \text{etc.}\right)$$

vbi ſi loco dv et δv valores modo inuenti ſubſtituantur erit

$$dx\delta v - dv\delta x = dx\int dx\left(\mathfrak{N}'\omega + \frac{\mathfrak{P}' d\omega}{dx} + \frac{\mathfrak{Q}' dd\omega}{dx^2} + \frac{\mathfrak{R}' d^3\omega}{dx^3} + \text{etc.}\right)$$

Nunc

Nunc ponatur $\int L dx = I$ integrali ita ſumto vt euaneſcat in integrationis initio, in fine autem fiat $I = A$, et habebimus

$$\int L(dx\delta v - dv\delta x) = \int (A-I) dx (\mathfrak{N}'\omega + \frac{\mathfrak{P}'d\omega}{dx} + \frac{\mathfrak{Q}'dd\omega}{dx^2} + \frac{\mathfrak{R}'d^3\omega}{dx^3} + \text{etc.})$$

Reſtituantur pro $\mathfrak{N}'$, $\mathfrak{P}'$, $\mathfrak{Q}'$, $\mathfrak{R}'$ etc. valores ſupra aſſumti et ad calculum contrahendum ponatur;

$$N + (A-I)\mathfrak{N} + (A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{n} = N'$$
$$P + (A-I)\mathfrak{P} + (A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{p} = P'$$
$$Q + (A-I)\mathfrak{Q} + (A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{q} = Q'$$
$$R + (A-I)\mathfrak{R} + (A-I)(\mathfrak{A}-I)\mathfrak{r} = R'$$

etc.

ac manifeſtum eſt fore variationem quaeſitam:

$$\delta\int V dx = V\delta x + \int dx (N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.})$$

quae forma porro euoluitur in eandem expreſſionem quam ſub finem probl 9. (110.) exhibuimus, quam ergo hic denuo apponere foret ſuperfluum.

Coroll. 1.

120. Hic ergo formula integralis $\int V dx$, cuius variationem aſſignauimus ita eſt comparata, vt non ſolum functio V formulam integralem $\int \mathfrak{V} dx$ inuoluat, ſed etiam haec functio $\mathfrak{V}$ aliam formulam integralem $\int \mathfrak{v} dx$ in ſe complectatur; vbi quidem functio $\mathfrak{v}$ nullam amplius formulam integralem implicat.

Coroll. 2.

121. Sin autem et haec functio $\mathfrak{v}$ insuper formulam integralem in se inuoluat, iam satis perspicuum est, quomodo tum solutionem institui oporteat; siquidem tum valores N', P', Q', R' etc. partes insuper recipient, a postrema formula integrali pendentes.

Scholion.

122. Quomodocunque ergo formula integralis $\int V dx$ fuerit complicata, praecepta hactenus exposita omnino sufficiunt ad eius variationem inuestigandam etiamsi forte complicatio fuerit infinita. Cum igitur omnes expressiones binas variabiles implicantes, quarum variationes vnquam sint inuestigandae vel a formulis integralibus sint liberae, vel vnam pluresue in se complectantur, easque vel simplices vel complicatas vtcunque, huic Calculi variationum parti, quae circa duas variabiles versatur, abunde satisfactum videtur vt vix quicquam amplius desiderari queat. Quamobrem ad formulas trium variabilium progrediamur ac primo quidem tales, quarum relatio per geminam aequationem definiri ponitur, vt binae variabiles tanquam functiones tertiae spectari queant, siue haec duplex relatio sit cognita siue ex ipsa variationis indole inuestiganda.

CAPVT V.

DE VARIATIONE FORMVLARVM INTEGRALIVM VARIABILES INVOLVENTIVM, ET DVPLICEM RELATIONEM IMPLICANTIVM.

Problema 12.

133.

Propofita formula quacunque ternas variabiles x, y, z cum fuis differentialibus cuiuscunque gradus inuoluente, eius variationem defiuire ex variationibus omnium trium variabilium oriundam.

Solutio.

Sit W formula ifta propofita, cuius primo quaeratur valor variatus $W + \delta W$, qui oritur fi loco x, y, z fcribantur ipfarum valores variati

$$x + \delta x,\ y + \delta y,\ z + \delta z,$$

fimiliterque pro earum differentialibus

$$dx + d\delta x,\ dy + d\delta y,\ dz + d\delta z$$

et ita porro: a quo fi ipfa formula W auferatur

 rema-

remanebit eius variatio δW. Ex quo intelligitur hanc variationem per consuetam differentiationem obtineri si modo loco signi differentiationis d, signum variationis δ scribatur. Tantum notasse iuvabit, si differentialium variationes capi oporteat, perinde esse, in quonam loco inter differentiationis signa signum variationis δ collocetur, quemadmodum supra demonstrauimus; vnde signum variationis perpetuo in postremo loco poni poterit, quod cum ad formulas integrales progrediemur, commodissimum videtur, sicut ex iis quae hactenus de formulis integralibus binas variabiles inuoluentibus, sunt tradita, satis est manifestum.

Coroll. 1.

124. Quoniam z perinde ac y tanquam functio ipsius x spectari potest, si ponatur

$\frac{dy}{dx}=p$ et $\frac{dz}{dx}=\mathfrak{p}$, erit

$\delta p=\frac{d\delta y-pd\delta x}{dx}$ et $\delta\mathfrak{p}=\frac{d\delta z-\mathfrak{p}d\delta x}{dx}$,

similique modo formulae hinc deriuatae a superioribus non discrepant.

Coroll. 2.

125. Ponamus

$\delta y-p\delta x=\omega$ et $\delta z-\mathfrak{p}\delta x=\mathfrak{w}$,

eritque

$d\delta y-pd\delta x-qdx\delta x=d\omega$ et $d\delta z-\mathfrak{p}d\delta x-\mathfrak{q}dx\delta x=d\mathfrak{w}$,

si

si scilicet statuamus

$$\frac{dp}{dx}=q \quad \text{et} \quad \frac{d\mathfrak{p}}{dx}=\mathfrak{q},$$

vnde patet fore

$$\delta p-q\delta x=\frac{d\omega}{dx} \quad \text{et} \quad \delta\mathfrak{p}-\mathfrak{q}\delta x=\frac{d\mathfrak{w}}{dx}.$$

Coroll. 3.

126. Si vlterius statuamus

$$\frac{dq}{dx}=r;\ \frac{d\mathfrak{q}}{dx}=\mathfrak{r};\ \frac{dr}{dx}=s;\ \frac{d\mathfrak{r}}{dx}=\mathfrak{s} \text{ etc.}$$

erit simili modo sumto dx constante

$$\delta q-r\delta x=\frac{dd\omega}{dx^2};\ \delta\mathfrak{q}-\mathfrak{r}\delta x=\frac{dd\mathfrak{w}}{dx^2}$$

$$\delta r-s\delta x=\frac{d^3\omega}{dx^3};\ \delta\mathfrak{r}-\mathfrak{s}\delta x=\frac{d^3\mathfrak{w}}{dx^3}$$

sicque deinceps.

Scholion 1.

127. Siue ergo formula varianda habuerit valorem finitum siue infinitum, siue euanescentem ope horum praeceptorum eius variatio perinde ac supra inueniri potest, neque enim haec praecepta a superioribus discrepant, nisi quod hic duplicis generis valores differentiales, alteri litteris latinis p, q, r, s etc. alteri germanicis $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ etc. indicati introduci debeant, cuius rei ratio in eo est sita quod hic vtraque variabilis y et z tanquam functio ipsius x spectari potest. Sin autem vnica aequatio inter ternas coordinatas daretur vel quaereretur, litterae hic introductae p et $\mathfrak{p}$ nullos habiturae essent valores

res certos, cum salua illa aequatione fractiones $\frac{dy}{dx}$ et $\frac{dz}{dx}$ omnes omnino valores recipere possent. Omissis autem his litteris ipsisque differentialibus in calculo relictis, etiam pro hoc casu regula in solutione exposita variationem declarabit.

Scholion 2.

128. Supra iam notaui hunc casum trium variabilium quarum relatio gemina aequatione definitur, sollicite esse distinguendum ab eo, vbi relatio vnica aequatione definiri assumitur. Discrimen hoc ex Geometria clarissime illustratur, vbi ternae variabiles vicem ternarum coordinatarum gerunt; totidem autem in calculo adhiberi oportet non solum quando quaestio circa superficies versatur, sed etiam quando lineae curuae non in eodem plano sitae sunt explorandae. Atque hoc quidem casu posteriori determinatio lineae curuae duas aequationes inter ternas coordinatas postulat, ita vt binae quaeuis tanquam functiones tertiae spectari possint. Superficiei autem natura iam vnica aequatione inter ternas coordinatas definitur, ita vt vnaquaeque tanquam functio binarum reliquarum spectari queat, vnde ingens discrimen in ipsa tractatione oritur. Praesens igitur caput inseruire poterit eiusmodi lineis curuis indagandis quae non in eodem plano sitae maximi minimiue quapiam gaudeant proprietate.

Proble-

Problema 13.

129. Si V fuerit functio quaecunque trium variabilium x, y, z earum insuper differentialia cuiusque ordinis implicans eaeque variabiles variationes quascunque recipiant, inuenire variationem formulae integralis $\int V dx$.

Solutio.

Quaecunque differentialia in functionem V ingrediantur ea his factis substitutionibus:

$$dy = p dx;\ dp = q dx;\ dq = r dx;\ dr = s dx \text{ etc.}$$

$$dz = \mathfrak{p} dx;\ d\mathfrak{p} = \mathfrak{q} dx;\ d\mathfrak{q} = \mathfrak{r} dx;\ d\mathfrak{r} = \mathfrak{s} dx \text{ etc.}$$

tollentur, et quantitas V erit functio quantitatum finitarum x, y, z, p, q, r, s etc. $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ etc. Eius ergo differentiale huiusmodi habebit formam:

$$dV = M dx + N dy + P dp + Q dq + R dr + S ds + \text{ etc.}$$
$$+ \mathfrak{N} dz + \mathfrak{P} d\mathfrak{p} + \mathfrak{Q} d\mathfrak{q} + \mathfrak{R} d\mathfrak{r} + \mathfrak{S} d\mathfrak{s} + \text{ etc.}$$

vnde mutatis signis differentiationis d in δ simul habebitur variatio δV. Ex supra autem demonstratis etiam pro hoc casu trium variabilium habebitur

$$\delta \int V dx = \int (V d\delta x + dx \delta V) = V \delta x + \int (dx \delta V - dV \delta x).$$

At facta substitutione fiet

$$\frac{dx\delta V - dV\delta x}{dx} = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}$$
$$+ \mathfrak{N}\delta z + \mathfrak{P}\delta \mathfrak{p} + \mathfrak{Q}\delta \mathfrak{q} + \mathfrak{R}\delta \mathfrak{r} + \text{etc.}$$
$$- M\delta x - Np\delta x - Pq\delta x - Qr\delta x - Rs\delta x - \text{etc.}$$
$$- \mathfrak{N}\mathfrak{p}\delta x - \mathfrak{P}\mathfrak{q}\delta x - \mathfrak{Q}\mathfrak{r}\delta x - \mathfrak{R}\mathfrak{s}\delta x - \text{etc.}$$

Quodsi iam breuitatis gratia statuamus

$$\delta y - p\delta x = \omega \text{ et } \delta z - \mathfrak{p}\delta x = \mathfrak{w}$$

sumto elemento dx constante, ex §. §. 125. et 126. erit

$$\delta p - q\delta x = \frac{d\omega}{dx} \; ; \; \delta \mathfrak{p} - \mathfrak{q}\delta x = \frac{d\mathfrak{w}}{dx}$$
$$\delta q - r\delta x = \frac{dd\omega}{dx^2} \; ; \; \delta \mathfrak{q} - \mathfrak{r}\delta x = \frac{dd\mathfrak{w}}{dx^2}$$
$$\delta r - s\delta x = \frac{d^3\omega}{dx^3} \; ; \; \delta \mathfrak{r} - \mathfrak{s}\delta x = \frac{d^3\mathfrak{w}}{dx^3}$$

etc.

vnde variatio quaesita hoc modo commode exprimetur

$$\delta\int V dx = V\delta x + \int dx \left\{ \begin{array}{l} N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \text{etc.} \\ \mathfrak{N}\mathfrak{w} + \frac{\mathfrak{P}d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}d^3\mathfrak{w}}{dx^3} + \text{etc.} \end{array} \right\}$$

quae

quae vt ſupra ad hanc formam reducitur:

$$\begin{aligned}\delta\int V\,dx = &+\int \omega\,dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right)\\ &+\int \mathfrak{w}\,dx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right)\\ &+V\delta x + \omega\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right)\\ &+\text{Conſt.} + \mathfrak{w}\left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \frac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right)\\ &\qquad + \frac{d\omega}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right)\\ &\qquad + \frac{d\mathfrak{w}}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \frac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right)\\ &\qquad + \frac{dd\omega}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right)\\ &\qquad + \frac{dd\mathfrak{w}}{dx^2}\left(\mathfrak{R} - \frac{d\mathfrak{S}}{dx} + \text{etc.}\right)\\ &\qquad + \frac{d^3\omega}{dx^3}\left(S - \text{etc.}\right)\\ &\qquad + \frac{d^3\mathfrak{w}}{dx^3}\left(\mathfrak{S} - \text{etc.}\right)\end{aligned}$$

cuius indoles ex ſuperioribus ſatis eſt manifeſta, eademque circa conſtantis additionem ſunt obſeruanda.

Coroll. 1.

130. In hac ſolutione ambae variabiles y et z tanquam functiones ipſius x ſpectantur, ſiue iam ſint cognitae ſiue demum ex variationis indole definiendae. Neque etiam formula integralis $\int V\,dx$ certum eſſet habitura valorem, niſi tam y quam z per x determinari conciperetur.

Coroll. 2.

131. Si formula $V\,dx$ per ſe ſit integrabilis, nulla aſſumta relatione inter ternas variabiles, variatio integralis $\int V\,dx$ nullas quoque formulas integrales inuoluere poteſt, ideoque neceſſe eſt vt tum ſit:

$$\text{et } N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

$$\text{et } \mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.} = 0.$$

Coroll. 3.

132. Viciſſim etiam ſi hae duae aequationes locum habeant, hoc certum erit criterium formulam differentialem $V\,dx$ per ſe integrationem admittere, nulla inter variabiles ſtabilita relatione.

Exemplum.

133. *Quo hoc criterium magis illuſtremus ſumamus eiusmodi formulam per ſe integrabilem, ſitque*

$$\int V\,dx = \frac{z\,dy}{x\,dz} = \frac{pz}{xp},$$

vnde ſit

$$V = \frac{-pz}{xxp} + \frac{p}{x} + \frac{zq}{xp} - \frac{zpq}{xpp}.$$

Ex cuius differentiatione colligimus $N = 0$, et

$$P = \frac{-z}{xxp} + \frac{1}{x} - \frac{zq}{xpp}; \quad Q = \frac{z}{xp} \text{ porro}$$

$$\mathfrak{N} = \frac{-p}{xxp} + \frac{q}{xp} - \frac{pq}{xpp}.$$

$$\mathfrak{P} = \frac{pz}{xxpp} - \frac{zq}{xpp} + \frac{2zpq}{xp^3} \text{ et } \mathfrak{Q} = \frac{-zp}{xpp}.$$

Iam

Iam pro prima aequatione ob $N=0$ fieri oportet

$$-\frac{dP}{dx}+\frac{ddQ}{dx^2}=0 \text{ seu } P-\frac{dQ}{dx}=\text{Const.}$$

cuius veritas ex differentiatione ipsius Q statim fit perspicua.

Pro altera aequatione

$$\mathfrak{R}-\frac{d\mathfrak{P}}{dx}+\frac{dd\mathfrak{Q}}{dx^2}=0,$$

quia hinc est

$$\int\mathfrak{R}\,dx=\mathfrak{P}-\frac{d\mathfrak{Q}}{dx},$$

primo necesse est vt integrabilis existat haec formula

$$\mathfrak{R}\,dx=\frac{-p\,dx}{xx\mathfrak{p}}+\frac{q\,dx}{x\mathfrak{p}}-\frac{p\,dx}{x\mathfrak{p}\mathfrak{p}},$$

vnde ob $q\,dx=dp$ manifesto fit

$$\int\mathfrak{R}\,dx=\frac{p}{x\mathfrak{p}}.$$

Superest ergo vt fit

$$\frac{d\mathfrak{Q}}{dx}=\mathfrak{P}-\int\mathfrak{R}\,dx=\frac{pz}{xx\mathfrak{p}\mathfrak{p}}-\frac{zq}{x\mathfrak{p}\mathfrak{p}}+\frac{2zpq}{x\mathfrak{p}^3}-\frac{p}{x\mathfrak{p}}.$$

Verum differentiando $\mathfrak{Q}=\frac{-zp}{x\mathfrak{p}\mathfrak{p}}$, vtrinque perfecta aequalitas resultat.

Scholion 1.

134. Quodsi ergo quaestio huc redeat, vt formulae integrali $\int V\,dx$ valor maximus minimusue sit conciliandus, tum ante omnia in eius variatione ambas partes integrales idque seorsim nihilo aequari oportet, propterea quod vtcunque variationes con-

ſtituantur, variatio $\delta\int V\,dx$ ſemper debeat euaneſcere, vnde duae emergunt aequationes iſtae

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0 \quad \text{et}$$

$$\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.} = 0$$

quibus duplex relatio inter ternas variabiles x, y, z ita exprimitur, vt deinceps tam y quam z recte tanquam functio ipſius x ſpectari poſſit. Quando autem hae aequationes ſunt differentiales idque altioris gradus, totidem vtrinque conſtantes arbitrariae per integrationes in calculum inuehuntur, quoti gradus vtraque fuerit differentialis. Has vero conſtantes deinceps ita definiri oportet, vt conditionibus tam pro initio quam pro fine integrationis formulae $\int V\,dx$ praeſcriptis ſatisfiat quod negotium eo redit, vt praeterea variationis partes abſolutae ad nihilum redigantur. Primo ſcilicet *conſtans* ita definiri debet, vt conditionibus pro initio praeſcriptis ſatisfiat, vbi quidem ex quaeſtionis indole particulae

$$\omega,\ \mathfrak{w},\ \frac{d\omega}{dx},\ \frac{d\mathfrak{w}}{dx},\ \frac{dd\omega}{dx^2},\ \frac{dd\mathfrak{w}}{dx^2}\ \text{etc.}$$

definitos valores ſortiri ſolent. Tum vero cum idem circa finem integrationis vſu veniat, ex ſingulis conſtantes per integrationem ingreſſae determinabuntur.

Scholion 2.

135. Plurimum conducet hic obſeruaſſe membra, quibus variatio $\delta\int V\,dx$ exprimitur, ſponte in duas claſſes diſpeſci, in quarum altera litterae tantam eae con-

conſpiciuntur, quae ad variabilitatem ipſius y, ſeu ad eius habitum reſpectu x referuntur, idque ita ac ſi quantitas z conſtans eſſet aſſumta, altera vero claſſis ſimiles literas a variabilitate ipſius z tantum pendentes, continet, quaſi quantitas y eſſet conſtans. Ex quo colligere licet, ſi etiam quarta variabilis v accedat, quae vt functio ipſius x quoque ſpectari queat, tum ad illas duas claſſes tertiam inſuper eſſe adiiciendam, quae ſimilia membra a variabilitate ſolius v pendentia complectatur. Quocirca ſolutio hic data ſpectari poteſt, quaſi ad quotcunque variabiles extendatur, dummodo tot inter eas aequationes dari concipiantur, vt omnes pro functionibus vnius haberi queant. Etſi ergo hoc caput tantum tres variabiles prae ſe fert, tamen ad quotcunque pertinere eſt intelligendum, ſi modo eiusmodi conditiones proponantur, vt tandem per vnam reliquae omnes determinentur. Talem autem conditionem formulae integrales huius formae $\int V\,dx$ neceſſario inuoluunt; quotcunque enim variabiles in quantitatem V ingrediantur, expreſſio $\int V\,dx$ certum valorem definitum omnino obtinere nequit, niſi omnes variabiles tanquam functiones vnius x ſpectari queant. Longe aliter autem eſt comparata ratio earum formularum integralium, quae ad duas pluresves variabiles a ſe inuicem minime pendentes referuntur.

Proble-

Problema 14.

136. Si functio V praeter tres variabiles x, y, z earumque differentialia cuiucunque gradus insuper inuoluat formulam integralem $v = \int \mathfrak{V}dx$, vbi $\mathfrak{V}$ sit functio quaecunque earundem variabilium x, y, z cum suis differentialibus inuestigare variationem formulae integralis $\int Vdx$.

Solutio.

Vt species saltem differentialium e calculo tollatur, ponamus vt ante:

$$dy = pdx, \; dp = qdx, \; dq = rdx, \; dr = sdx \text{ etc.}$$
$$dz = \mathfrak{p}dx, \; d\mathfrak{p} = \mathfrak{q}dx, \; d\mathfrak{q} = \mathfrak{r}dx, \; d\mathfrak{r} = \mathfrak{s}dx \text{ etc.}$$

ac functione V differentiata prodeat

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + Rdr + \text{ etc.}$$
$$+ \mathfrak{N}dz + \mathfrak{P}d\mathfrak{p} + \mathfrak{Q}d\mathfrak{q} + \mathfrak{R}d\mathfrak{r} + \text{ etc.}$$

tum vero ob $dv = \mathfrak{V}dx$ sit

$$d\mathfrak{V} = M'dx + N'dy + P'dp + Q'dq + R'dr + \text{ etc.}$$
$$+ \mathfrak{N}'dz + \mathfrak{P}'d\mathfrak{p} + \mathfrak{Q}'d\mathfrak{q} + \mathfrak{R}'d\mathfrak{r} + \text{ etc.}$$

vbi ob defectum litterarum iisdem accentu distinctis vtor. Hinc autem simul earundem quantitatum V et $\mathfrak{V}$ variationes habentur. Iam cum quaeratur variatio $\delta \int Vdx$, habebimus primo quidem vt ante

$$\delta \int Vdx = V\delta x + \int (dx\delta V - dV\delta x)$$

vbi cum valor ipsius V non discrepet a praecedente, nisi

nisi quod hic ad eius differentiale dV accedat pars $L dv = L \mathfrak{V} dx$ et ad variationem δV haec pars $L \delta v = L \delta \int \mathfrak{V} dx$; etiam variatio quaesita $\delta \int V dx$ forma ante inuenta exprimetur, si modo ad eam adiiciatur hoc membrum:

$$\int L(dx \delta \int \mathfrak{V} dx - \mathfrak{V} dx \delta x) = \int L dx(\delta \int \mathfrak{V} dx - \mathfrak{V} \delta x).$$

Quia vero formula integralis $\int \mathfrak{V} dx$ eadem est quae in problemate praecedente est tractata, si vt ibi fecimus, statuamus

$$\delta y - p \delta x = \omega \text{ et } \delta z - \mathfrak{p} \delta x = \mathfrak{w},$$

elemento dx constante assumto habebimus

$$\delta \int \mathfrak{V} dx - \mathfrak{V} \delta x = \int dx \begin{cases} N' \omega + \frac{P' d\omega}{dx} + \frac{Q' dd\omega}{dx^2} + \frac{R' d^3 \omega}{dx^3} + \text{etc.} \\ \mathfrak{N}' \mathfrak{w} + \frac{\mathfrak{P}' d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}' dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}' d^3 \mathfrak{w}}{dx^3} + \text{etc.} \end{cases}$$

Ponamus iam integrale $\int L dx = I$ si scilicet ita capiatur, vt pro initio integrationis euanescat, tum vero pro termino finali integrationis fiat $I = A$, quo facto pro tota integrationis extensione erit

$$\int L dx(\delta \int \mathfrak{V} dx - \mathfrak{V} \delta x) = \int (A - I) dx \begin{cases} N' \omega + \frac{P' d\omega}{dx} + \frac{Q' dd\omega}{dx^2} + \text{etc.} \\ \mathfrak{N}' \mathfrak{w} + \frac{\mathfrak{P}' d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}' dd\mathfrak{w}}{dx^2} + \text{etc.} \end{cases}$$

Nunc igitur introducamus sequentes abbreuiationes

$$\begin{array}{ll} N + (A - I) N' = N^\circ ; & \mathfrak{N} + (A - I) \mathfrak{N}' = \mathfrak{N}^\circ \\ P + (A - I) P' = P^\circ ; & \mathfrak{P} + (A - I) \mathfrak{P}' = \mathfrak{P}^\circ \\ Q + (A - I) Q' = Q^\circ ; & \mathfrak{Q} + (A - I) \mathfrak{Q}' = \mathfrak{Q}^\circ \\ R + (A - I) R' = R^\circ ; & \mathfrak{R} + (A - I) \mathfrak{R}' = \mathfrak{R}^\circ \\ \text{etc.} & \text{etc.} \end{array}$$

 atque

atque manifestum est variationem quaesitam ita expressam iri:

$$\delta\int V dx = V\delta x + \int dx \begin{cases} N^\circ\omega + \frac{P^\circ d\omega}{dx} + \frac{Q^\circ dd\omega}{dx^2} + \frac{R^\circ d^3\omega}{dx^3} + \text{etc.} \\ \mathfrak{N}^\circ\mathfrak{w} + \frac{\mathfrak{P}^\circ d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}^\circ dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}^\circ d^3\mathfrak{w}}{dx^3} + \text{etc.} \end{cases}$$

quae etiam vt ante euoluitur in hanc formam:

$$\begin{aligned} \delta\int V dx = & + \int\omega dx\left(N^\circ - \frac{dP^\circ}{dx} + \frac{ddQ^\circ}{dx^2} - \frac{d^3R^\circ}{dx^3} + \frac{d^4S^\circ}{dx^4} - \text{etc.}\right) \\ & + \int\mathfrak{w} dx\left(\mathfrak{N}^\circ - \frac{d\mathfrak{P}^\circ}{dx} + \frac{dd\mathfrak{Q}^\circ}{dx^2} - \frac{d^3\mathfrak{R}^\circ}{dx^3} + \frac{d^4\mathfrak{S}^\circ}{dx^4} - \text{etc.}\right) \\ & + V\delta x + \omega\left(P^\circ - \frac{dQ^\circ}{dx} + \frac{ddR^\circ}{dx^2} - \frac{d^3S^\circ}{dx^3} + \text{etc.}\right) \\ & + \text{Const.} + \mathfrak{w}\left(\mathfrak{P}^\circ - \frac{d\mathfrak{Q}^\circ}{dx} + \frac{dd\mathfrak{R}^\circ}{dx^2} - \frac{d^3\mathfrak{S}^\circ}{dx^3} + \text{etc.}\right) \\ & + \frac{d\omega}{dx}\left(Q^\circ - \frac{dR^\circ}{dx} + \frac{ddS^\circ}{dx^2} - \text{etc.}\right) \\ & + \frac{d\mathfrak{w}}{dx}\left(\mathfrak{Q}^\circ - \frac{d\mathfrak{R}^\circ}{dx} + \frac{dd\mathfrak{S}^\circ}{dx^2} - \text{etc.}\right) \\ & + \frac{dd\omega}{dx^2}\left(R^\circ - \frac{dS^\circ}{dx} + \text{etc.}\right) \\ & + \frac{dd\mathfrak{w}}{dx^2}\left(\mathfrak{R}^\circ - \frac{d\mathfrak{S}^\circ}{dx} + \text{etc.}\right) \\ & + \frac{d^3\omega}{dx^3}\left(S^\circ - \text{etc.}\right) \\ & + \frac{d^3\mathfrak{w}}{dx^3}\left(\mathfrak{S}^\circ - \text{etc.}\right) \end{aligned}$$

vbi neminem offendat signum nihili litteris suffixum siquidem non exponentem denotat, sed tantum ad has litteras ab iisdem nude positis distinguendas adhibetur.

Coroll. 1.

Coroll. 1.

137. Si igitur formula integralis $\int V dx$ habere debeat valorem maximum vel minimum, variationis inuentae bina membra priora ſtatim nihilo aequalia ſtatui oportet, vnde duae reſultant aequationes differentiales, quibus indefinita relatio vtriusque variabilis y et z ad x definitur.

Coroll. 2.

138. Etiamſi hic conditionum, quae forte pro initio et fine integrationis proponantur, nondum ratio habetur, tamen ea iam occulte in calculum ingreditur, quia litterae I et A terminos integrationis reſpiciunt. Interim tamen eae in ipſa aequationum differentialium tractatione iterum ex calculo expelluntur; dum enim formula integralis $\int L dx = I$ eliditur, ſimul quantitas conſtans A egreditur.

Coroll. 3.

139. Expeditis autem aequationibus his duabus differentialibus, idque generaliſſime, vt totidem conſtantes arbitrariae in calculum inuehantur, quot integrationes inſtitui oportuit, tum demum ad conditiones vtriusque termini integrationis formulae $\int V dx$ eſt attendendum, quandoquidem hinc ex reliquis variationis membris abſolutis illae conſtantes determinari debent.

Scholion.

140. Solutio huius problematis ita eſt comparata vt iam ſatis ſit perſpicuum, quemadmodum etiam formulas magis complicatas, veluti ſi functio V plures formulas integrales inuoluat, vel ſi quoque functio 𝔙 formulas nouas integrales complectatur, expediri conueniat. Quin etiam nunc eſt manifeſtum, ſi huiusmodi formulae integrales plures tribus variabiles contineant, quomodo tum variationes inueniri oporteat, atque adeo non ſolum taedioſum ſed etiam ſuperfluum foret ſi copioſius hoc argumentum perſequi vellem. Ad partem igitur huius doctrinae alteram multo abſtruſiorem progredior, vbi etiam relationibus inter variabiles conſtitutis duae pluresue a ſe inuicem minime pendentes in calculo relinquuntur.

CAPVT

CAPVT VI.

DE VARIATIONE FORMVLARVM DIFFERENTIALIVM TRES VARIABILES INVOLVENTIVM, QVARVM RELATIO VNICA AEQVATIONE CONTINETVR.

Problema 15.

141.

Propofita aequatione inter tres variabiles x, y et z, quibus variationes quaecunque δx, δy, δz tribuuntur, definire variationes formularum differentialium primi gradus:

$$p = \left(\frac{dz}{dx}\right) \text{ et } p' = \left(\frac{dz}{dy}\right).$$

Solutio.

Cum vnica aequatio inter tres variabiles dari ponitur, quaelibet earum tanquam functio binarum reliquarum fpectari poteft. Erit ergo z functio ipfarum x et y, et meminiffe hic oportet expreffionem $\left(\frac{dz}{dx}\right) = p$ denotare rationem differentialium ipfarum z et x, fi in aequatione illa data hae folae

 vt

vt variabiles tractentur, tertia y pro constante habita, quod idem de altera formula $p' = (\frac{dz}{dy})$ est tenendum. Simili modo ipsae quoque variationes δx, δy, δz vt functiones infinite paruae binarum variabilium x et y spectari possunt, quoniam si etiam a tertia z penderent, haec ipsa est functio ipsarum x et y; vnde simul intelligitur quid istae formulae

$$(\frac{d\delta z}{dx});\ (\frac{d\delta z}{dy}),\ \text{item}$$

$$(\frac{d\delta x}{dx});\ (\frac{d\delta x}{dy})\ \text{et}\ (\frac{d\delta y}{dx});\ (\frac{d\delta y}{dy})$$

significent. Cum igitur valor variatus formulae

$$(\frac{dz}{dx}) = p \ \text{sit}\ p + \delta p = (\frac{d(z+\delta z)}{d(x+\delta x)}),$$

si scilicet hic variabilis y constans sumatur, erit hac conditione obseruata

$$p + \delta p = (\frac{dz + d\delta z}{dx + d\delta x}) = (\frac{dz}{dx} + \frac{d\delta z}{dx} - \frac{dz\, d\delta x}{dx^2})$$

propterea quod variationes δx et δz prae x et z euanescunt. Hinc ergo ob $(\frac{dz}{dx}) = p$ habebitur variatio quaesita:

$$\delta p = (\frac{d\delta z}{dx}) - (\frac{dz}{dx} \cdot \frac{d\delta x}{dx}) = (\frac{d\delta z}{dx}) - p(\frac{d\delta x}{dx}),$$

quarum formularum significatus, cum tam δz quam δx sint functiones ipsarum x et y, hicque y constans habeatur, per se est manifestus. Simili autem modo reperietur fore

$$\delta p' = (\frac{d\delta z}{dy}) - p'(\frac{d\delta y}{dy}),$$

vbi iam variabilis x pro constante habetur.

Coroll. 1.

Coroll. 1.

142. Hic omnia ad binas variabiles x et y sunt perducta, atque vt earum functiones spectantur, non solum tertia z, sed etiam omnes tres variationes δx, δy, δz, manifestum autem est, has tres variabiles pro lubitu inter se permutari posse.

Coroll. 2.

143. Sufficit autem his binis formulis pro differentialibus primi gradus vti, quoniam reliquas ad has reducere licet, siquidem sit

$$\left(\frac{dx}{dz}\right)=\frac{1}{p};\ \left(\frac{dy}{dz}\right)=\frac{1}{p'},\ \text{et}$$
$$\left(\frac{dy}{dx}\right)=\frac{-p}{p'};\ \text{et}\ \left(\frac{dx}{dy}\right)=\frac{-p'}{p},$$

vbi p et p' sunt functiones binarum x et y.

Coroll. 3.

144. Inuentis ergo variationibus harum duarum formularum

$$p=\left(\frac{dz}{dx}\right)\ \text{et}\ p'=\left(\frac{dz}{dy}\right),$$

reliquarum formularum modo memoratarum variationes hinc facile reperientur. Erit enim:

$$\delta\left(\frac{dx}{dz}\right)=-\frac{\delta p}{pp}=-\frac{1}{pp}\left(\frac{d\delta z}{dx}\right)+\frac{1}{p}\left(\frac{d\delta x}{dx}\right)$$
$$\delta\left(\frac{dy}{dz}\right)=-\frac{\delta p'}{p'p'}=-\frac{1}{p'p'}\left(\frac{d\delta z}{dy}\right)+\frac{1}{p'}\left(\frac{d\delta y}{dy}\right)$$
$$\delta\left(\frac{dy}{dx}\right)=-\frac{\delta p}{p'}+\frac{p\,\delta p'}{p'p'}=-\frac{1}{p'}\left(\frac{d\delta z}{dx}\right)+\frac{p}{p'}\left(\frac{d\delta x}{dx}\right)+\frac{p}{p'p'}\left(\frac{d\delta z}{dy}\right)$$
$$-\frac{p}{p'}\left(\frac{d\delta y}{dy}\right).$$

Scho-

Scholion I.

145. Hic ante omnia obſeruo formulas differentiales certum valorem habere non poſſe, niſi duo differentialia ita inter ſe comparentur, vt tertia variabilis, ſi tres habeantur ſeu reliquae omnes, ſi plures adſint, conſtantes accipiantur. Ita hoc caſu quo inter tres variabiles x, y et z vnica aequatio datur, vel ſaltem dari concipitur, formula $\frac{dz}{dx}$ nullum plane habet ſignificatum, niſi tertia variabilis y conſtans ſumatur, quam conditionem vinculis includendo hanc formulam innuere conſueuerunt, etiamſi ea tuto omitti poſſent, quoniam alioquin ne vllus quidem ſignificatus adeſſet. Quod quo magis perſpicuum reddatur, quaecunque aequatio inter ternas variabiles x, y, z proponatur, ex ea valor ipſius z elici concipiatur, vt z aequetur certae functioni ipſarum x et y, eiusque ſumto differentiali prodeat $dz = pdx + p'dy$, vbi iterum p et p' certae erunt functiones ipſarum x et y, idque tales vt ſit $(\frac{dp}{dy}) = (\frac{dp'}{dx})$. Sumta nunc y conſtante fit $dz = pdx$ ſeu $p = (\frac{dz}{dx})$, ſumta autem x conſtante prodit $p' = (\frac{dz}{dy})$. Tum vero etiam manifeſtum eſt ſumta z conſtante fore $\frac{dy}{dx} = \frac{-p}{p'}$, huiusmodi autem formulas excludi conueniet, quando tam z quam variationes δx, δy et δz vt functiones ipſarum x et y repraeſentamus.

Scholion 2.

146. Ex Geometria hoc argumentum multo clarius illuſtrare licet. Denotent enim tres noſtrae variabiles x, y, z ternas coordinatas AX, XY, YZ, Fig. 4. inter quas aequatio propoſita certam quandam ſuperficiem aſſignabit, in qua ordinata $YZ = z$ terminabitur, quae vtique tanquam certa functio binarum reliquarum $AX = x$ et $XY = y$ ſpectari poteſt, ita vt ſumtis pro lubitu his binis x et y, tertia $YZ = z$ ex aequatione propoſita determinetur. Quodſi iam alia ſuperficies quaecunque concipiatur ab iſta infinite parum diſcrepans, eaque ita cum hac comparetur, vt eius punctum quoduis z cum propoſitae puncto Z conferatur, ita tamen vt interuallum Zz ſit ſemper infinite paruum, variationes ita repraeſentabuntur, vt ſit

$$\delta x = Ax - AX = Xx\,;\ \delta y = xy - XY \text{ et } \delta z = yz - YZ;$$

et cum hae variationes prorſus arbitrio noſtro permittantur, neque vllo modo a ſe inuicem pendeant, eae etiam tanquam functiones binarum x et y ſpectari poſſunt, idque ita vt nulla a reliquis pendeat, ſed vnaquaeque pro arbitrio fingi queat. Quin etiam hinc intelligitur, quoniam ſuperficies proxima a propoſita diuerſa eſſe debet, neutiquam fore

$$\delta z = p\,\delta x + p'\,\delta y,$$

ſiquidem pro ſuperficie propoſita fuerit

$$dz = p\,dx + p'\,dy;$$

alioquin punctum z foret in eadem superficie, ex quo omnino ternas functiones ipsarum x et y pro variationibus δx, δy et δz ita comparatas esse oportet, vt non sit

$$\delta z = p\delta x + p'\delta y$$

sed potius ab hoc valore quomodocunque discrepet; vbi quidem imprimis notandum est; has functiones ita late patere, vt discontinuae non excludantur, atque adeo pro lubitu variationes tantum in vnico puncto vel saltem exiguo spatio constitui queant. Ne autem hic vlli dubio locus relinquatur, probe notandum est; ex eo quod ponimus z eiusmodi functionem ipsarum x et y, vt sit

$$dz = p\,dx + p'\,dy,$$

minime sequi fore quoque

$$\delta z = p\delta x + p'\delta y,$$

quemadmodum supra assumsimus, propterea quod hic ipsi z propriam tribuimus variationem neutiquam pendentem a variationibus ipsarum x et y.

Problema 16.

147. Proposita aequatione inter tres variabiles x, y, z, quibus variationes quaecunque δx, δy, δz tribuuntur, inuestigare variationes formularum differentialium secundi gradus:

$$q = \left(\frac{ddz}{dx^2}\right);\ q' = \left(\frac{ddz}{dx\,dy}\right) \text{ et } q'' = \left(\frac{ddz}{dy^2}\right).$$

Solutio.

Solutio.

Hic iterum z ſpectatur vt functio ipſarum x et y, quarum etiam ſunt functiones ternae variationes δx, δy, δz nullo modo a ſe inuicem pendentes. Quoniam in praecedente problemate poſuimus

$$p=\left(\frac{dz}{dx}\right) \text{ et } p'=\left(\frac{dz}{dy}\right),$$

his formulis in ſubſidium vocatis habebimus

$$q=\left(\frac{dp}{dx}\right);\; q'=\left(\frac{dp}{dy}\right)=\left(\frac{dp'}{dx}\right) \text{ et } q''=\left(\frac{dp'}{dy}\right),$$

hicque ratio variationum δp et $\delta p'$ eſt habenda quas inuenimus:

$$\delta p=\left(\frac{d\delta z}{dx}\right)-p\left(\frac{d\delta x}{dx}\right) \text{ et } \delta p'=\left(\frac{d\delta z}{dy}\right)-p'\left(\frac{d\delta y}{dy}\right).$$

Simili ergo modo calculum ſubducendo reperiemus primo:

$$\delta q=\left(\frac{d\delta p}{dx}\right)-q\left(\frac{d\delta x}{dx}\right);$$

vbi $\left(\frac{d\delta p}{dx}\right)$ inuenitur ſi valor δp differentietur poſita y conſtante, ac differentiale per dx diuidatur, vnde oritur:

$$\left(\frac{d\delta p}{dx}\right)=\left(\frac{dd\delta z}{dx^2}\right)-q\left(\frac{d\delta x}{dx}\right)-p\left(\frac{dd\delta x}{dx^2}\right) \text{ ob } q=\left(\frac{dp}{dx}\right)$$

vnde concludimus:

$$\delta q=\left(\frac{dd\delta z}{dx^2}\right)-2q\left(\frac{d\delta x}{dx}\right)-p\left(\frac{dd\delta x}{dx^2}\right).$$

Eodem modo ob $q'=\left(\frac{dp}{dy}\right)$ erit

$$\delta q'=\left(\frac{d\delta p}{dy}\right)-q'\left(\frac{d\delta y}{dy}\right) \text{ at}$$

$$\left(\frac{d\delta p}{dy}\right)=\left(\frac{dd\delta z}{dx\,dy}\right)-q'\left(\frac{d\delta x}{dx}\right)-p\left(\frac{dd\delta x}{dx\,dy}\right),$$

ideoque

$$\delta q' = \left(\frac{dd\delta z}{dx\,dy}\right) - q'\left(\frac{d\delta x}{dx}\right) - q'\left(\frac{d\delta y}{dy}\right) - p\left(\frac{dd\delta x}{dx\,dy}\right).$$

Alter autem valor $q' = \left(\frac{dp'}{dx}\right)$ ſimili modo tractatus praebet:

$$\delta q' = \left(\frac{dd\delta z}{dx\,dy}\right) - q'\left(\frac{d\delta x}{dx}\right) - q'\left(\frac{d\delta y}{dy}\right) - p'\left(\frac{dd\delta y}{dx\,dy}\right)$$

cuius valoris ab illo diſcrepantia incommodum involuit mox accuratius examinandum. Ex tertia autem formula $q'' = \left(\frac{dp'}{dy}\right)$ elicitur:

$$\delta q'' = \left(\frac{dd\delta z}{dy^2}\right) - 2q''\left(\frac{d\delta y}{dy}\right) - p'\left(\frac{dd\delta y}{dy^2}\right).$$

Scholion. 1.

148. In originem diſcrepantiae variationis $\delta q'$, ex gemino valore

$$q' = \left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right)$$

natae inquiſiturus, obſeruo in his formulis variationem exprimentibus, vel quantitatem x vel quantitatem y pro conſtanti haberi, prout denominator cuiuscunque membri declarat. Verum ſi quantitatem x conſtantem manere ſumimus, vtcunque interea altera y mutabilis exiſtit, natura rei poſtulat, vt etiam variationes ipſius x nullam mutationem ſubeant, quod autem ſecus euenit, ſi variatio δx quoque a quantitate y pendeat, quod idem de altera variabili y, dum conſtans ponitur, eſt tenendum. Ex quo manifeſtum eſt, ſi variationes δx et δy ſimul ab ambabus variabilibus x et y pendere ſumantur,

mantur, id ipsi hypothesi, qua alterutra perpetuo constans ponitur aduersari. Quamobrem hoc incommodum aliter vitari nequit nisi statuamus, variationem ipsius x prorsus non ab altera variabili y, neque huius variationem δy ab altera x pendere. Sin autem δx per solam x, et δy per solam y determinatur, vt sit et

$$\left(\frac{d\delta x}{dy}\right)=0 \text{ et } \left(\frac{d\delta y}{dx}\right)=0$$

erit etiam

$$\left(\frac{dd\delta x}{dx\,dy}\right)=0 \text{ et } \left(\frac{dd\delta y}{dx\,dy}\right)=0$$

sicque ambo illi valores discrepantes pro $\delta q'$ inuenti ad consensum perducuntur.

Scholion 2.

149. Omnibus autem dubiis in hac inuestigatione felicissime occurremus, si soli quantitati z variationes tribuamus, binis reliquis x et y plane inuariatis relictis, ita vt sit tam $\delta x=0$ quam $\delta y=0$, quo pacto non solum calculo consulitur, sed etiam vsus huius calculi variationum vix restringitur. Quodsi enim superficiem quamcunque cum alia sibi proxima comparamus, nihil impedit, quominus singula proposita superficiei puncta ad ea proximae puncta referamus, quibus eaedem binae coordinatae x et y respondeant, solaque tertia z variationem patiatur. Quin etiam haec suppositio, cum ad formulas integrales progrediemur eo magis est necessaria

ria quandoquidem semper totum negotium ad eiusmodi formulas integrales perducitur, quae duplicem integrationem requirunt, in quarum altera sola x in altera vero sola y vt variabilis tractatur; nisi ergo harum variationes nullae statuantur, maxima incommoda inde in calculum inueherentur; qui cum per se plerumque sit difficillimus minime consultum videtur, vt ex hac parte difficultates multiplicentur. Quamobrem hanc tractationem ita sum expediturus, vt in posterum perpetuo binis variabilibus x et y nullas plane variationes tribuam solamque tertiam z variatione quacunque δz augeri assumam, vbi quidem δz vt functionem quamcunque ipsarum x et y siue continuam siue discontinuam sum spectaturus.

Problema 17.

150. Si z fuerit functio quaecunque ipsarum x et y, eique tribuatur variatio δz pariter vtcunque ab x et y pendens, inuestigare variationes formularum omnium differentialium cuiuscunque ordinis.

Solutio.

Pro differentialibus primi gradus habentur hae duae formulae

$$p = \left(\frac{dz}{dx}\right) \text{ et } p' = \left(\frac{dz}{dy}\right),$$

quarum vatiationes cum x et y nullam variationem pati

pati concipiantur, ex supra inuentis ita habebunt:

$$\delta p = \left(\frac{d\delta z}{dx}\right) \text{ et } \delta p' = \left(\frac{d\delta z}{dy}\right).$$

Pro differentialibus secundi ordinis, hae tres formulae habentur:

$$q = \left(\frac{ddz}{dx^2}\right);\; q' = \left(\frac{ddz}{dx\,dy}\right) \text{ et } q'' = \left(\frac{ddz}{dy^2}\right)$$

ita vt sit

$$q = \left(\frac{dp}{dx}\right);\; q' = \left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right) \text{ et } q'' = \left(\frac{dp'}{dy}\right)$$

quarum variationes ex praecedente problemate ob $\delta x = 0$ et $\delta y = 0$ sunt:

$$\delta q = \left(\frac{dd\delta z}{dx^2}\right);\; \delta q'\left(\frac{dd\delta z}{dx\,dy}\right);\; \delta q''\left(\frac{dd\delta z}{dy^2}\right).$$

Simili modo si ad differentialia tertii ordinis ascendamus, hae quatuor formulae occurrunt:

$$r = \left(\frac{d^3 z}{dx^3}\right);\; r' = \left(\frac{d^3 z}{dx^2\,dy}\right);\; r'' = \left(\frac{d^3 z}{dx\,dy^2}\right);\; r''' = \left(\frac{d^3 z}{dy^3}\right)$$

quarum variationes ita expressum iri manifestum est:

$$\delta r = \left(\frac{d^3\delta z}{dx^3}\right);\; \delta r' = \left(\frac{d^3\delta z}{dx^2\,dy}\right);\; \delta r'' = \left(\frac{d^3\delta z}{dx\,dy^2}\right);$$
$$\delta r''' = \left(\frac{d^3\delta z}{dy^3}\right)$$

vnde per se patet, quomodo variationes formularum differentialium superiorum ordinum sint exprimendae.

Coroll. 1.

151. Hinc iam manifestum est fore in genere pro formula differentiali cuiuscunque ordinis $\left(\frac{d^{\mu+\nu} z}{dx^\mu\, dy^\nu}\right)$

eius

eius variationem $=\left(\frac{d^{\mu+\nu}\delta z}{dx^{\mu}dy^{\nu}}\right)$, in qua forma superiores omnes continentur.

Coroll. 2.

152. Deinde etiam perspicuum est introducendis loco differentialium primi ordinis litteris p, p', secundi ordinis litteris q, q', q'', tertii ordinis litteris r, r', r'', r''', quarti ordinis litteris s, s', s'', s''', s^{IV} etc. speciem differentialium tolli, quemadmodum etiam supra huiusmodi litteris speciem differentialium sustulimus.

Scholion.

153. Quoniam binae variabiles x et y prorsus a se inuicem non pendent, ita vt altera adeo eundem valorem retinere queat, dum altera per omnes valores possibiles variatur, euidens huiusmodi formulam differentialem $\frac{dy}{dx}$, quippe quae nullum plane significatum certum esset habitura, in calculo nunquam locum inuenire posse. Contra vero cum quantitas z sit functio ipsarum x et y, hae formulae $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ et reliquae omnes quas supra sum contemplatus, definitos habent significatus, neque vllae aliae in calculum ingredi possunt. Deinde quia semper quaestiones huc pertinentes eo reducere licet, vt z tanquam functio binarum x et y spectari possit, eiusmodi formulas $\left(\frac{dy}{dx}\right)$ vbi quantitas z

esset

effet pro conftanti habita hinc prorfus excluduntur, neque vllae aliae praeter fupra memoratas in. calculo admitti funt cenfendae, ficque omnes expreffiones a formulis integralibus liberae practer ipfas variabiles x, y, z alias formulas differentiales non implicabunt praeter eas, quarum variationes hic funt indicatae.

Problema 18.

154. Si z fit functio ipfarum x et y, eique tribuatur variatio δz vtcunque ab x et y pendens, tum vero fuerit V quantitas quomodocunque ex tribus variabilibus x, y, z earumque differentialibus cuiuscunque ordinis compofita, eius variationem δV inueftigare.

Solutio.

Vt in expreffione V fpecies differentialium tollantur, ponamus vt hactenus fecimus:

$$p = \left(\frac{dz}{dx}\right); \quad p' = \left(\frac{dz}{dy}\right)$$

$$q = \left(\frac{ddz}{dx^2}\right); \quad q' = \left(\frac{ddz}{dx\,dy}\right); \quad q'' = \left(\frac{ddz}{dy^2}\right)$$

$$r = \left(\frac{d^3z}{dx^3}\right); \quad r' = \left(\frac{d^3z}{dx^2\,dy}\right); \quad r'' = \left(\frac{d^3z}{dx\,dy^2}\right); \quad r''' = \left(\frac{d^3z}{dy^3}\right)$$

etc.

quarum formularum variationes a variatione ipfius z oriundas ita definimus, vt pofita euidentiae gratia ifta variatione $\delta z = \omega$, quam vt functionem quamcunque binarum variabilium x et y spectari oportet, fit

$$\delta p = \left(\frac{d\omega}{dx}\right);\ \delta p' = \left(\frac{d\omega}{dy}\right)$$

$$\delta q = \left(\frac{dd\omega}{dx^2}\right);\ \delta q' = \left(\frac{dd\omega}{dx\,dy}\right);\ \delta q'' = \left(\frac{dd\omega}{dy^2}\right)$$

$$\delta r = \left(\frac{d^3\omega}{dx^3}\right);\ \delta r' = \left(\frac{d^3\omega}{dx^2 dy}\right);\ \delta r'' = \left(\frac{d^3\omega}{dx\,dy^2}\right);\ \delta r''' = \left(\frac{d^3\omega}{dy^3}\right)$$

etc.

Illis autem factis substitutionibus expressio proposita V fiet functio harum quantitatum x, y, z, p, p', q, q', q'', r, r', r'', r''' etc. Eius ergo differentiale talem induet formam:

$$\begin{aligned} dV = L\,dx + M\,dy + N\,dz + P\,dp &+ Q\,dq + R\,dr \\ &+ P'dp' + Q'dq' + R'dr' \\ &\qquad + Q''dq'' + R''dr'' \\ &\qquad\qquad + R'''dr''' \end{aligned}$$

etc.

Quoniam nunc formula V eatenus tantum variationem recipit, quatenus quantitates, ex quibus componitur, variantur binae autem x et y immunes statuuntur, eius variatio quam quaerimus erit:

$$\begin{aligned} \delta V = N\,\delta z + P\,\delta p &+ Q\,\delta q + R\,\delta r \\ &+ P'\delta p' + Q'\delta q' + R'\delta r' \\ &\qquad + Q''\delta q'' + R''\delta r'' \\ &\qquad\qquad + R'''\delta r''' \end{aligned}$$

etc.

ac ſi loco variationis δz ſcribamus ω habebimus variationes inuentas ſubſtituendo:

$$\begin{aligned}\delta V = N\omega + P\left(\tfrac{d\omega}{dx}\right) + Q\left(\tfrac{dd\omega}{dx^2}\right) + R\left(\tfrac{d^3\omega}{dx^3}\right) \\ + P'\left(\tfrac{d\omega}{dy}\right) + Q'\left(\tfrac{dd\omega}{dx\,dy}\right) + R'\left(\tfrac{d^3\omega}{dx^2\,dy}\right) \\ + Q''\left(\tfrac{dd\omega}{dy^2}\right) + R''\left(\tfrac{d^3\omega}{dx\,dy^2}\right) \\ + R'''\left(\tfrac{d^3\omega}{dy^3}\right)\end{aligned}$$

etc.

cuius formatio, ſi forte etiam differentialia altiorum graduum ingrediantur; per ſe eſt manifeſta.

Coroll. 1.

155. Cum ω ſpectetur vt functio binarum variabilium x et y, ſingularum partium, quae variationem δV conſtituunt, ſignificatus eſt determinatus, atque haec variatio perfecte definita eſt cenſenda.

Coroll. 2.

156. Quomodocunque autem expreſſio V differentialibus ſit referta, quandoquidem valorem certum indicare eſt cenſenda, ſubſtitutionibus adhibitis ſemper a ſpecie differentialium liberari debet.

Coroll. 3.

157. Si noſtrae tres variabiles ad ſuperficiem referantur vt ſint eius coordinatae $AX = x$, $XY = y$, Fig. 6.
$YZ = z$, ſola ordinata $YZ = z$ vbique incremen-

 tum

tum infinite paruum $Zz = \delta z = \omega$ accipere intelligitur, ita vt puncta z cadant in aliam superficiem ab illa infinite parum discrepantem.

Scholion.

158. Dubio hic occurri debet inde oriundo, quod quantitatem z vt functionem binarum x et y spectandam esse diximus; quoniam enim ipsis x et y nullas variationes tribuimus, si in expressione V loco z eius valor in x et y substitueretur, ea ipsa in meram functionem ipsarum x et y abiret, neque propterea vllam variationem esset receptura. Verum notandum est, tametsi z vt functio ipsarum x et y consideratur, eam tamen plerumque esse incognitam, quando scilicet eius naturam demum ex conditione variationis erui oportet; sin autem iam ab initio esset data, tamen dum variatio quaeritur, functionem hanc z quasi incognitam spectari conuenit, minimeque eius loco valorem per x et y expressum substitui licet, antequam variatio, quippe quae a sola z pendet, penitus fuerit explorata.

CAPVT

CAPVT VII.

DE VARIATIONE FORMVLARVM INTEGRALIVM TRES VARIABILES INVOLVENTIVM, QVARVM VNA VT FVNCTIO BINARVM RELIQVARVM SPECTATVR.

Problema 19.

159.

Formularum integralium huc pertinentium naturam euoluere, ac rationem qua earum variationes inuestigari conueniat, explicare.

Solutio.

Cum tres habeantur variabiles x, y et z, quarum vna z vt functio binarum reliquarum x et y est spectanda, etiamsi in ipsa variationis inuestigatione ratio huius functionis pro incognita haberi debet; formulae integrales quae in hoc calculi genere occurrunt, plurimum discrepant ab iis, quae circa binas tantum variabiles proponi solent. Quemadmodum enim talis forma integralis $\int V dx$, vbi V duas variabiles x et y implicare censetur, quarum y ab x pendere concipitur, quasi summa omnium

 valo-

valorum elementarium $V dx$ per omnes valores ipſius x collectorum conſiderari poteſt; ita quando tres variabiles x, y et z habentur, quarum haec z a binis x et y ſimul pendere concipitur, integralia huc pertinentia collectionem omnium elementorum ad omnes valores tam ipſius x, quam ipſius y relatorum inuoluunt, ideoque duplicem integrationem alteram per omnes valores ipſius x, alteram vero ipſius y elementa congregantem requirunt. Ex quo huiusmodi integralia tali forma $\int\int V dx dy$ contineri debent, qua ſcilicet duplex integratio innuatur; cuius euolutio ita inſtitui ſolet, vt primo altera variabilis y vt conſtans ſpectetur, et formulae $\int V dx$ valor per terminos integrationis extenſus quaeratur; in quo cum iam x obtineat valorem vel datum vel ab y pendentem, hoc integrale $\int V dx$ in functionem ipſius y tantum abibit, qua in dy ducta ſupereſt vt integrale $\int dy \int V dx$ inueſtigetur, quae ergo forma $\int dy \int V \delta x$ hoc modo tractata illi $\int\int V dx dy$ aequiualere eſt cenſenda. Ac ſi ordine inuerſo primo quantitas x conſtans accipiatur, et integrale $\int V dy$ per terminos praeſcriptos extendatur, id deinceps vt functio ipſius x ſpectari et integrale quaeſitum $\int dx \int V dy$ inueniri poterit. Perinde autem eſt vtro modo valorem integralis formulae duplicatae $\int\int V dx dy$ vtamur.

Cum igitur in hoc genere aliae formulae integrales niſi huiusmodi $\int\int V dx dy$ occurrere nequeant, totum

totum negotium huc redit, vt quemadmodum huiusmodi formae variationem inueniri oporteat, ostendamus. Quoniam autem quantitates x et y variationis expertes assumimus, ex iis quae initio sunt demonstrata facile colligitur fore

$$\delta \int\int V\, dx\, dy = \int\int \delta V\, dx\, dy,$$

vbi δV variationem ipsius V denotat; hicque integratione pariter duplici est opus, prorsus vt modo ante innuimus.

Coroll. 1.

160. Si ponamus integrale $\int\int V\, dx\, dy = W$, cum sit $\int dx \int V\, dy = W$ erit per solam x differentiando $\int V\, dy = (\frac{dW}{dx})$ hincque porro per y differentiando $V = (\frac{ddW}{dx\, dy})$, vnde patet integrale W ita comparatum esse vt fiat $V = (\frac{ddW}{dx\, dy})$.

Coroll. 2.

161. Cum duplex integratio sit instituenda, vtraque quantitas arbitraria introducitur; altera autem integratio loco constantis functionem quamcunque ipsius x quae sit X, altera autem functionem quamcunque ipsius y, quae sit Y inuehit, ita vt completum integrale sit

$$\int\int V\, dx\, dy = W + X + Y.$$

Coroll. 3.

Coroll. 3.

162. Hoc etiam per ipsam resolutionem confirmatur, fit enim primo

$$\int V\,dy = \left(\frac{dW}{dx}\right) + \left(\frac{dX}{dx}\right) \text{ ob } \left(\frac{dY}{dy}\right) = 0.$$

Tum vero fit $V = \left(\frac{ddW}{dx\,dy}\right)$ quia neque X neque $\frac{dX}{dx}$ ab y pendet. Quare si fuerit $\left(\frac{ddW}{dx\,dy}\right) = V$, erit integrale completum

$$\int\int V\,dx\,dy = W + X + Y.$$

Scholion 1.

163. Omnino autem necessarium est, vt indoles huiusmodi formularum integralium duplicatarum $\int\int V\,dx\,dy$ accuratius examini subiicietur, quod commodissime per Theoriam superficierum praestari poterit. Sint ergo vt hactenus x et y binae coordinatae orthogonales in basi assumtae $AX = x$,
Fig. 7. $XY = y$, cui in Y normaliter insistat tertia ordinata $YZ = z$ ad superficiem vsque porrecta. Si iam binae illae coordinatae x et y suis differentialibus crescant $XX' = dx$ et $YY' = dy$, inde in basi oritur parallelogrammum elementare $YxyY' = dx\,dy$, cui elementum formulae integralis conuenit. Ita si de soliditate a superficie inclusa sit quaestio eius elementum erit $= z\,dx\,dy$, ideoque tota soliditas $= \int\int z\,dx\,dy$; si superficies ipsa quaeratur, posito $dz = p\,dx + p'\,dy$ erit eius elementum huic rectangulo $dx\,dy$ imminens $= dx\,dy\,\sqrt{(1 + pp + p'p')}$, ideo-

ideoque ipsa superficies $= \int\int dx\,dy\, V(1+pp+p'p')$, ex quo generatim intelligitur ratio formulae integralis duplicatae $\int\int V\,dx\,dy$. Quod si iam talis formulae valor quaeratur, qui dato spatio in basi veluti ADYX respondeat, primo sumta x constante inuestigetur integrale simplex $\int V\,dy$, ac tum ipsi y assignetur magnitudo XY ad curuam DY porrecta, quae ex huius curuae natura aequabitur certae functioni ipsius x. Sic igitur $dx\int V\,dy$ exprimet formulae propositae elementum rectangulo $XYxX' = y\,dx$ conueniens, cuius integrale denuo sumtum $\int dx \int V\,dy$ et ex sola variabili x constans tandem dabit valorem toti spatio ADYX respondentem, siquidem vtraque integratio adiectione constantis rite determinetur.

Scholion 2.

164. Ita se habere debet euolutio huiusmodi formularum integralium duplicatarum, si ad figuram in basi datam veluti ADYX fuerit accommodanda; sin autem vtramque integrationem indefinite expedire velimus, vt primo sumta x constante quaeramus integrale $\int V\,dy$, quod rectangulo elementari $XYyX' = y\,dx$ conuenire est intelligendum, siquidem in dx ducatur, deinde vero in integratione formulae $\int dx \int V\,dy$ quantitatem $y = XY$ eandem manere concipiamus, sola x pro variabili sumta, tum valor prodibit rectangulo indefinito $APYX = xy$ respondens, si quidem constantes per vtramque in-

tegrationem ingressae debite definiantur. At si spatii istius reliqui termini praeter lineas XY et PY vt indefiniti spectentur, integrale $\int\int V\,dx\,dy$ recipiet binas functiones X+Y indefinitas illam ipsius x, hanc vero ipsius y. Quodsi ergo ad calculum maximorum et minimorum haec deinceps accommodare velimus, quoniam maximi minimiue proprietas, quae in spatium quodpiam datum ADYX competere debet, simulquoque cuiuis spatio indefinito veluti APYX conueniat necesse est, duplicem illam integrationem modo hic exposito indefinito administrari conueniet.

Problema 20.

165. Si V sit formula quaecunque ex ternis variabilibus x, y, z earumque differentialibus composita, inuenire variationem formulae integralis duplicatae $\int\int V\,dx\,dy$, dum quantitati z, quae vt functio binarum x et y spectatur, variationes quaecunque tribuuntur.

Solutio.

Ad speciem differentialium tollendam statuamus:

$$p=\left(\frac{dz}{dx}\right);\ p'=\left(\frac{dz}{dy}\right)$$

$$q=\left(\frac{dp}{dx}\right);\ q'=\left(\frac{dp}{dy}\right)=\left(\frac{dp'}{dx}\right);\ q''=\left(\frac{dp'}{dy}\right)$$

$$r=\left(\frac{dq}{dx}\right);\ r'=\left(\frac{dq}{dy}\right)=\left(\frac{dq'}{dx}\right);\ r''=\left(\frac{dq'}{dy}\right)=\left(\frac{dq''}{dx}\right);\ r'''=\left(\frac{dq''}{dy}\right)$$

etc.

vt

vt V fiat functio quantitatum finitarum $x, y, z, p, p', q, q', q'', r, r', r'', r'''$ etc. Tum ponatur eius differentiale:

$$dV = L\,dx + M\,dy + N\,dz + P\,dp + Q\,dq + R\,dr + P'\,dp' + Q'\,dq' + R'\,dr' + Q''\,dq'' + R''\,dr'' + R'''\,dr''' \text{ etc.}$$

ex quo cum simul eius variatio δV innotescat, ex problemate praecedente colligitur variatio quaesita

$$\delta \int\!\!\int V\,dx\,dy = \int\!\!\int dx\,dy(N\,\delta z + P\,\delta p + Q\,\delta q + R\,\delta r + P'\,\delta p' + Q'\,\delta q' + R'\,\delta r' + Q''\,\delta q'' + R''\,\delta r'' + R'''\,\delta r''' \text{ etc.}$$

Quodsi iam vti §. 154. fecimus ponamus variationem $\delta z = \omega$, quam vt functionem quamcunque binarum variabilium x et y spectare licet; indidem istam variationem concludimus fore:

$$\delta \int\!\!\int V\,dx\,dy = \int\!\!\int dx\,dy\Big(N\omega + P\left(\tfrac{d\omega}{dx}\right) + Q\left(\tfrac{dd\omega}{dx^2}\right) + R\left(\tfrac{d^3\omega}{dx^3}\right) + P'\left(\tfrac{d\omega}{dy}\right) + Q'\left(\tfrac{dd\omega}{dx\,dy}\right) + R'\left(\tfrac{d^3\omega}{dx^2 dy}\right) + Q''\left(\tfrac{dd\omega}{dy^2}\right) + R''\left(\tfrac{d^3\omega}{dx\,dy^2}\right) + R'''\left(\tfrac{d^3\omega}{dy^3}\right) \text{ etc.}$$

 Coroll. 1.

Coroll. 1.

166. Si ergo vtriusque functionis z et $\delta z = \omega$ indoles, seu ratio compositionis ex binis variabilibus x et y esset data, tum per praecepta ante exposita variatio formulae integralis duplicatae $\int\int V\,dxdy$ assignari posset; quomodocunque quantitas V ex variabilibus x, y, z earumque differentialibus fuerit conflata.

Coroll. 2.

167. Totum scilicet negotium redibit ad euolutionem formulae integralis duplicatae inuentae, quae cum pluribus constet partibus, singulas partes per duplicem integrationem, vti ante explicatum, tractari conueniet.

Scholion.

168. Quando autem ratio functionis z non constat, eaque demum ex conditione variationis elici debet, ita vt ipsa variatio $\delta z = \omega$ nullam plane derminationem patiatur, quemadmodum fit si formula $\int\int V\,dx\,dy$ valorem maximum minimumue obtinere debeat; tum omnino necessarium est, vt singula variationis inuentae $\delta\int\int V\,dx\,dy$ membra ita reducantur, vt vbique post signum integrationis duplicatum non valores differentiales variationis $\delta z = \omega$ sed haec ipsa variatio occurrat; cuiusmodi reductione iam supra in formulis binas tantum variabiles inuoluentibus sumus vsi. Talis autem reductio

ductio, cum pro formulis integralibus duplicatis minus ſit conſueta, accuratiorem explicationem poſtulat. Quem in finem obſeruo huiusmodi reductione perueniri ad formulas ſimpliciter integrales, in quibus alterutra tantum quantitatum x et y pro variabili habeatur, altera vt conſtante ſpectata, ad quod indicandum, ne ſigna praeter neceſſitatem multiplicentur, talis forma $\int T dx$ denotabit integrale formulae differentialis $T dx$, dum quantitas y pro conſtanti habetur; ſimilique modo intelligendum eſt in hac forma $\int T dy$ ſolam quantitatem y vt variabilem conſiderari quod eo magis perſpicuum eſt, cum hac conditione omiſſa, hae formulae nullum plane ſignificatum eſſent habiturae. Neque ergo in poſterum opus eſt declarari, ſi T ambas variabiles x et y complectatur, vtra earum in formulis integralibus ſimplicibus $\int T dx$ vel $\int T dy$, ſiue conſtans ſiue variabilis accipiatur, cum ea ſola, cuius differentiale exprimitur pro variabili ſit habenda. In formulis autem duplicatis $\iint V dx dy$ perpetuo tenendum eſt, alteram integrationem ad ſolius x, alteram vero ad ſolius y variabilitatem adſtringi, perindeque eſſe, vtra integratio prior inſtituatur.

Problema 21.

169. Variationem formulae integralis duplicatae $\iint V dx dy$ in praecedente problemate inuentam, ita transformare, vt poſt ſignum integrale duplicatum vbique ipſa variatio $\delta z = \omega$ occurrat, exturbatis eius differentialibus.

Solutio.

Quo haec transformatio latius pateat, sint T et v functiones quaecunque binarum variabilium x et y, et consideretur haec formula duplicata $\int\int T\,dx\,dy\left(\frac{dv}{dx}\right)$ quae separata integrationum varietate ita repraesentetur $\int dy \int T\,dx\left(\frac{dv}{dx}\right)$, vt in integratione $\int T\,dx\left(\frac{dv}{dx}\right)$ sola quantitas x vt variabilis spectetur. Tum autem erit $dx\left(\frac{dv}{dx}\right)=dv$, quia y pro constante habetur, ideoque fiet

$$\int T\,dv = Tv - \int v\,dT\,,$$

vbi cum in differentiali dT solius variabilis x ratio habetur, ad hoc declarandum loco dT scribi convenit $dx\left(\frac{dT}{dx}\right)$, ita vt sit

$$\int T\,dx\left(\frac{dv}{dx}\right) = Tv - \int v\,dx\left(\frac{dT}{dx}\right)$$

hincque nostra formula ita prodeat reducta:

$$\int\int T\,dx\,dy\left(\frac{dv}{dx}\right) = \int Tv\,dy - \int\int v\,dx\,dy\left(\frac{dT}{dx}\right).$$

Simili modo permutatis variabilibus consequemur:

$$\int\int T\,dx\,dy\left(\frac{dv}{dy}\right) = \int Tv\,dx - \int\int v\,dx\,dy\left(\frac{dT}{dy}\right).$$

Hoc iam quasi lemmate praemisso, variationis in praeced. probl. inuentae reductio ita se habebit:

$$\int\int P\,dx\,dy\left(\frac{d\omega}{dx}\right) = \int P\,\omega\,dy - \int\int \omega\,dx\,dy\left(\frac{dP}{dx}\right)$$

$$\int\int P'\,dx\,dy\left(\frac{d\omega}{dy}\right) = \int P'\,\omega\,dx - \int\int \omega\,dx\,dy\left(\frac{dP'}{dy}\right).$$

Porro pro sequentibus membris sit primo $\left(\frac{d\omega}{dx}\right)=v$ ideo-

ideoque $\left(\frac{dd\omega}{dx^2}\right)=\left(\frac{dv}{dx}\right)$, vnde colligitur:

$$\iint Q\,dx\,dy\left(\frac{dd\omega}{dx^2}\right)=\int Q\,dy\left(\frac{d\omega}{dx}\right)-\iint dx\,dy\left(\frac{dQ}{dx}\right)\left(\frac{d\omega}{dx}\right)$$

ac postremo membro similiter reducto, fit

$$\iint Q\,dx\,dy\left(\frac{dd\omega}{dx^2}\right)=\int Q\,dy\left(\frac{d\omega}{dx}\right)-\int\omega\,dy\left(\frac{dQ}{dx}\right)+\iint\omega\,dx\,dy\left(\frac{ddQ}{dx^2}\right).$$

Per eandem substitutionem habebimus $\left(\frac{dd\omega}{dx\,dy}\right)=\left(\frac{dv}{dy}\right)$ hincque

$$\iint Q'\,dx\,dy\left(\frac{dd\omega}{dx\,dy}\right)=\int Q'\,dx\left(\frac{d\omega}{dx}\right)-\iint dx\,dy\left(\frac{d\omega}{dx}\right)\left(\frac{dQ'}{dy}\right)$$ seu

$$\iint Q'\,dx\,dy\left(\frac{dd\omega}{dx\,dy}\right)=\int Q'\,dx\left(\frac{d\omega}{dx}\right)-\int\omega\,dy\left(\frac{dQ'}{dy}\right)+\iint\omega\,dx\,dy\left(\frac{ddQ'}{dx\,dy}\right)$$

quae forma ob

$$\int Q'\,dx\left(\frac{d\omega}{dx}\right)=Q'\omega-\int\omega\,dx\left(\frac{dQ'}{dx}\right)$$

abit in hanc

$$\iint Q'\,dx\,dy\left(\frac{dd\omega}{dx\,dy}\right)=Q'\omega-\int\omega\,dx\left(\frac{dQ'}{dx}\right)+\iint\omega\,dx\,dy\left(\frac{ddQ'}{dx\,dy}\right)-\int\omega\,dy\left(\frac{dQ'}{dx}\right)$$

tum vero pro tertia forma huius ordinis nanciscimur:

$$\iint Q''\,dx\,dy\left(\frac{dd\omega}{dy^2}\right)=\int Q''\,dx\left(\frac{d\omega}{dy}\right)-\int\omega\,dx\left(\frac{dQ''}{dy}\right)+\iint\omega\,dx\,dy\left(\frac{ddQ''}{dy^2}\right)$$

Porro ob $\left(\frac{d^3\omega}{dx^3}\right)=\left(\frac{ddv}{dx^2}\right)$ manente $v=\left(\frac{d\omega}{dx}\right)$, fiet

$$\iint R\,dx\,dy\left(\frac{ddv}{dx^2}\right)=\int R\,dy\left(\frac{dv}{dx}\right)-\int v\,dy\left(\frac{dR}{dx}\right)+\iint v\,dx\,dy\left(\frac{ddR}{dx^2}\right)$$ et

$$\iint v\,dx\,dy\left(\frac{ddR}{dx^2}\right)=\int\omega\,dy\left(\frac{ddR}{dx^2}\right)-\iint\omega\,dx\,dy\left(\frac{d^3R}{dx^3}\right)$$

ita

ita vt sit

$$\int\!\!\int R\,dx\,dy\left(\frac{d^3\omega}{dx^3}\right)=\int R\,dy\left(\frac{dd\omega}{dx^2}\right)-\int dy\left(\frac{d\omega}{dx}\right)\left(\frac{dR}{dx}\right)+\int\omega dy\left(\frac{ddR}{dx^2}\right)$$
$$-\int\!\!\int\omega\,dx\,dy\left(\frac{d^3R}{dx^3}\right).$$

Deinde ob $\left(\frac{d^3\omega}{dx^2dy}\right)=\left(\frac{ddv}{dx\,dy}\right)$ erit

$$\int\!\!\int R'dx\,dy\left(\frac{ddv}{dx\,dy}\right)=R'v-\int v\,dx\left(\frac{dR'}{dx}\right)+\int\!\!\int v\,dx\,dy\left(\frac{ddR'}{dx\,dy}\right)$$
$$-\int v\,dy\left(\frac{dR'}{dy}\right)$$

et quia hic

$$\int\!\!\int v\,dx\,dy\left(\frac{ddR'}{dx\,dy}\right)=\int\omega\,dy\left(\frac{ddR'}{dx\,dy}\right)-\int\!\!\int\omega\,dx\,dy\left(\frac{d^3R'}{dx^2dy}\right)$$

concludimus fore:

$$\int\!\!\int R'dxdy\left(\frac{d^3\omega}{dx^2dy}\right)=R'\left(\frac{d\omega}{dx}\right)-\int\left(\frac{d\omega}{dx}\right)dx\left(\frac{dR'}{dx}\right)+\int\omega\,dy\left(\frac{ddR'}{dx\,dy}\right)$$
$$-\int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR'}{dy}\right)-\int\!\!\int\omega dxdy\left(\frac{d^3R'}{dx^2dy}\right).$$

Tandem permutandis x et y hinc colligimus:

$$\int\!\!\int R''dxdy\left(\frac{d^3\omega}{dxdy^2}\right)=R''\left(\frac{d\omega}{dy}\right)-\int\left(\frac{d\omega}{dy}\right)dy\left(\frac{dR''}{dy}\right)+\int\omega\,dx\left(\frac{ddR''}{dxdy}\right)$$
$$-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR''}{dx}\right)-\int\!\!\int\omega dxdy\left(\frac{d^3R''}{dxdy^2}\right)$$

et

$$\int\!\!\int R'''dxdy\left(\frac{d^3\omega}{dy^3}\right)=\int R'''dx\left(\frac{dd\omega}{dy^2}\right)-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR''}{dy}\right)+\int\omega\,dx\left(\frac{ddR'''}{dy^2}\right)$$
$$-\int\!\!\int\omega dxdy\left(\frac{d^3R'''}{dy^3}\right).$$

Quos

Quos valores si substituamus reperimus:

$$\begin{aligned}
\delta \int\int V dx dy = \int\int \omega\, dx\, dy \Big(N & - \left(\tfrac{dP}{dx}\right) + \left(\tfrac{ddQ}{dx^2}\right) - \left(\tfrac{d^3 R}{dx^3}\right) \\
& - \left(\tfrac{dP'}{dy}\right) + \left(\tfrac{d\,dQ'}{dx\,dy}\right) - \left(\tfrac{d^3 R'}{dx^2 dy}\right) \\
& \qquad\qquad + \left(\tfrac{d\,dQ''}{dy^2}\right) - \left(\tfrac{d^3 R''}{dx\, dy^2}\right) \\
& \qquad\qquad\qquad\qquad - \left(\tfrac{d^3 R'''}{dy^3}\right) \\
& + \int P \omega\, dy + \int Q dy \left(\tfrac{d\omega}{dx}\right) - \int \omega\, dy \left(\tfrac{dQ}{dx}\right) + Q' \omega \\
& + \int P' \omega\, dx - \int \omega\, dx \left(\tfrac{dQ'}{dx}\right) - \int \omega\, dy \left(\tfrac{dQ'}{dy}\right) \\
& \qquad + \int Q'' dx \left(\tfrac{d\omega}{dy}\right) - \int \omega\, dx \left(\tfrac{dQ''}{dy}\right) \\
& + \int R dy \left(\tfrac{dd\omega}{dx^2}\right) + R' \left(\tfrac{d\omega}{dx}\right) - \int \left(\tfrac{d\omega}{dx}\right) dx \left(\tfrac{dR'}{dx}\right) - \int \left(\tfrac{d\omega}{dy}\right) dy \left(\tfrac{dR''}{dy}\right) + \int R''' dx \left(\tfrac{dd\omega}{dy^2}\right) \\
& - \int \left(\tfrac{d\omega}{dx}\right) dy \left(\tfrac{dR}{dx}\right) + R'' \left(\tfrac{d\omega}{dy}\right) - \int \left(\tfrac{d\omega}{dx}\right) dy \left(\tfrac{dR'}{dy}\right) - \int \left(\tfrac{d\omega}{dy}\right) dx \left(\tfrac{dR''}{dx}\right) - \int \left(\tfrac{d\omega}{dy}\right) dx \left(\tfrac{dR'''}{dy}\right) \\
& + \int \omega\, dy \left(\tfrac{ddR}{dx^2}\right) \qquad + \int \omega\, dy \left(\tfrac{ddR'}{dx\,dy}\right) + \int \omega\, dx \left(\tfrac{ddR''}{dx\,dy}\right) + \int \omega\, dx \left(\tfrac{ddR'''}{dy^2}\right).
\end{aligned}$$

Coroll. I.

170 Huius expressionis pars prima satis est perspicua, reliquae vero partes commode ita disponi possunt, vt earum ratio comprehendatur:

$$\left.\begin{array}{l} \int \omega dy \Big(P - \left(\tfrac{dQ}{dx}\right) + \left(\tfrac{ddR}{dx^2}\right) \\ \qquad - \left(\tfrac{dQ'}{dy}\right) + \left(\tfrac{ddR'}{dx\,dy}\right) \text{ etc.} \\ \qquad\quad + \left(\tfrac{ddR''}{dy^2}\right) \end{array}\right\} + \int \omega dx \Big(P' - \left(\tfrac{dQ''}{dy}\right) + \left(\tfrac{ddR'''}{dy^2}\right) \left.\begin{array}{l} \\ - \left(\tfrac{dQ'}{dx}\right) + \left(\tfrac{ddR''}{dx\,dy}\right) \text{etc.} \\ \quad + \left(\tfrac{ddR'}{dx^2}\right) \end{array}\right\}$$

$$\left.\begin{array}{l} + \int \left(\tfrac{d\omega}{dx}\right) dy \Big(Q - \left(\tfrac{dR}{dx}\right) \text{ etc.} \\ \qquad\qquad - \left(\tfrac{dR'}{dy}\right) \end{array}\right\} + \int \left(\tfrac{d\omega}{dy}\right) dx \Big(Q'' - \left(\tfrac{dR'''}{dy}\right) \text{etc.} \left.\begin{array}{l} \\ - \left(\tfrac{dR''}{dx}\right) \end{array}\right\}$$

$$+\int\left(\tfrac{dd\omega}{dx^2}\right)dy(\mathrm{R}-\text{etc.})\ +\int\left(\tfrac{dd\omega}{dy^2}\right)dx(\mathrm{R}'''-\text{etc.})$$

$$\left.\begin{matrix}+\omega\left(\mathrm{Q}'-\left(\tfrac{d\mathrm{R}'}{dx}\right)\text{etc.}\right.\\ -\left(\tfrac{d\mathrm{R}''}{dy}\right)\end{matrix}\right\}\ \begin{matrix}+\left(\tfrac{d\omega}{dx}\right)(\mathrm{R}'-\text{etc.})\\ +\left(\tfrac{d\omega}{dy}\right)(\mathrm{R}''-\text{etc.})\end{matrix}$$

Coroll. 2.

171. Hic leui attentione adhibita mox patebit, quomodo istae partes vlterius continuari debeant, si forte quantitas V differentialia altiorum graduum complectatur.

Coroll. 3.

172. In harum formularum integralium aliis, quae differentiali dy sunt affectae, quantitas x constans sumitur, cui tribuitur valor termino integrationis conueniens; aliis vero quae differentiali dx sunt affectae, y est constans et termino integrationis aequalis, vnde patet in terminis integrationum tam x quam y recipere valorem constantem.

Scholion I.

173. Haec ergo variationis formula ad eum casum est accommodata, quo vtriusque integrationis termini tribuunt tam ipsi x quam ipsi y valores constantes. Veluti si de superficie fuerit quaestio
Fig. 7. formula integralis $\int\int \mathrm{V}\,dx\,dy$ ad rectangulum APYX in basi assumtum est referenda; eiusque valor ita definiri debet, vt sumtis $x=0$ et $y=0$, qui sunt valores initiales, euanescat, quo facto statui oportet $x=\mathrm{AX}$

$x = AX$ et $y = AP$, qui ſunt valores finales; atque ad eandem legem ipſa variatio inuenta eſt expedienda. Quodſi iam ea quaeratur ſuperficies, in qua formulae $\int\int V\,dx\,dy$ hoc modo definitae valor fiat maximus vel minimus, ante omnia neceſſe eſt, vt pars variationis prima duplicem integrationem inuoluens, ad nihilum redigatur, quomodocunque variatio $\delta z = \omega$ accipiatur, vnde haec naſcetur aequatio:

$$\begin{aligned} o = N &- \left(\frac{dP}{dx}\right) + \left(\frac{ddQ}{dx^2}\right) - \left(\frac{d^3R}{dx^3}\right) + \text{etc.} \\ &- \left(\frac{dP'}{dy}\right) + \left(\frac{ddQ'}{dx\,dy}\right) - \left(\frac{d^3R'}{dx^2\,dy}\right) \\ &\phantom{- \left(\frac{dP'}{dy}\right)} + \left(\frac{ddQ''}{dy^2}\right) - \left(\frac{d^3R''}{dx\,dy^2}\right) \\ &\phantom{- \left(\frac{dP'}{dy}\right) + \left(\frac{ddQ''}{dy^2}\right)} - \left(\frac{d^3R'''}{dy^3}\right) \end{aligned}$$

qua natura ſuperficiei hac indole praeditae exprimetur. Conſtantes autem per duplicem integrationem ingreſſae ita determinari debent, vt reliquis variationis partibus ſatisfiat.

Scholion 2.

174. Quo haec inueſtigatio in ſe maxime abſtruſa exemplo illuſtretur, ponamus eiusmodi ſuperficiem inueſtigari debere, quae inter omnes alias eandem ſoliditatem includentes ſit minima. Hunc in finem efficiendum eſt vt haec formula integralis duplicata

$$\int\int dx\,dy\,(z + a\sqrt{(1 + pp + p'p')})$$

maximum minimumue euadat. Cum ergo ſit

$V = z + a\sqrt{(1 + pp + p'p')}$, erit

$L = 0$, $M = 0$, $N = 1$,

 atque

atque

$$P = \frac{ap}{\sqrt{(1+pp+p'p')}} \quad \text{et} \quad P' = \frac{ap'}{\sqrt{(1+pp+p'p')}},$$

ideoque

$$dV = N\,dz + P\,dp + P'\,dp'$$

existente

$$dz = p\,dx + p'\,dy.$$

Quare superficiei quaesitae natura hac aequatione exprimetur:

$$N - \left(\frac{dP}{dx}\right) - \left(\frac{dP'}{dy}\right) = 0 \quad \text{seu} \quad 1 = \left(\frac{dP}{dx}\right) + \left(\frac{dP'}{dy}\right).$$

Est vero:

$$\left(\frac{dP}{dx}\right) = \frac{a}{(1+pp+p'p')^{\frac{3}{2}}}\left((1+p'p')\left(\frac{dp}{dx}\right) - pp'\left(\frac{dp'}{dx}\right)\right)$$

$$\left(\frac{dP'}{dy}\right) = \frac{a}{(1+pp+p'p')^{\frac{3}{2}}}\left((1+pp)\left(\frac{dp'}{dy}\right) - pp'\left(\frac{dp}{dy}\right)\right)$$

vbi notetur esse $\left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right)$. Ex quo ista obtinetur aequatio:

$$\frac{(1+pp+p'p')^{\frac{3}{2}}}{a} = (1+p'p')\left(\frac{dp}{dx}\right) - 2pp'\left(\frac{dp}{dy}\right) + (1+pp)\left(\frac{dp'}{dy}\right)$$

quam autem quomodo tractari oporteat, haud patet, etiamsi facile perspiciatur in ea aequationem pro superficie sphaerica $zz = cc - xx - yy$, quin etiam cylindrica $zz = cc - yy$ contineri.

SVPPLEMENTVM,

CONTINENS

EVOLVTIONEM CASVVM SINGVLARIVM CIRCA INTEGRATIONEM

AEQVATIONVM DIFFERENTIALIVM.

EVOLVTIO
CASVVM PRORSVS SINGVLARIVM CIRCA INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM.

1.

Cum adhuc plurimae atque inter se maxime discrepantes methodi sint in medium allatae aequationes differentiales integrandi, quaestio exoritur summi sane momenti, an non vnica detur eaque aequabilis methodus, cuius ope omnes illas diuersas aequationes differentiales, quas etiamnum resoluere licuit, integrari queant? nullum enim est dubium quin inuentio talis methodi maxima incrementa in vniuersam Analysin esset allatura. Pluribus Geometris quidem separatio binarum variabilium

bilium huiusmodi methodum ſuppeditare eſt viſa, cum omnes aequationum differentialium integrationes vel hac ratione ſint integratae vel eo facile poſſint reuocari. Praeterquam autem quod haec methodus ſubſtitutionibus abſoluitur, quae plerumque non minorem ſagacitatem poſtulant, quam id ipſum quod quaeritur, ac nonnunquam ſoli caſui deberi videntur; haec methodus etiam neutiquam extenditur ad aequationes differentiales ſecundi altiorumque graduum; et qui tales aequationes adhuc tractauerunt, longe alia artificia in ſubſidium vocare ſunt coacti. Quamobrem ſeparationem variabilium nequaquam tanquam methodum vniformem ac latiſſime patentem ſpectare licet, quae omnes integrationes, quae adhuc ſucceſſerunt, in ſe complectatur.

2. Talem autem methodum vniuerſalem iam pridem mihi equidem indicaſſe videor, dum oſtendi propoſita quacunque aequatione differentiali ſiue primi ſiue altioris gradus, ſemper dari eiusmodi quantitatem, per quam ſi aequatio multiplicetur, euadat integrabilis, ita vt hoc modo nulla plane ſubſtitutione alibi anxie quaerenda ſit opus. Ex quo non dubito hanc methodum aequationes differentiales ope multiplicationem ad integrabilitatem reuocandi, tanquam latiſſime patentem atque naturae maxime conuenientem, pronunciare; cum nulla integratio adhuc ſit expedita, quae hoc modo non facile abſolui poſſit. Cum ſcilicet omnis aequatio diffe-

differentialis primi gradus in hac forma $Pdx+Qdy=0$ contineatur, denotantibus litteris P et Q functiones quascunque binarum variabilium x et y, semper datur eiusmodi multiplicator M, itidem functio quaedam ambarum variabilium x et y, vt facta multiplicatione haec forma $MPdx+MQdy$ fiat integrabilis; cuius propterea integrale quantitati constanti arbitrariae aequatum exhibebit aequationem integralem aequationis differentialis propositae $Pdx+Qdy=0$, quae eadem ratio quoque in aequationibus differentialibus altiorum graduum locum habet. Verum hoc argumentum hic fusius exponere non est animus; sed potius praestantiam huius methodi prae separatione variabilium etiam eiusmodi casibus quibus id minime videatur, simulque summam eius vtilitatem hic declarare constitui.

3. Quoties scilicet in aequatione differentiali variabiles x et y iam sunt separatae, totum negotium vulgo vt iam confectum spectari solet, quandoquidem huius aequationis

$$Xdx+Ydy=0,$$

vbi X denotat functionem solius x et Y solius y, integrale in promtu est

$$\int Xdx+\int Ydy=\text{Const.}$$

Interim tamen saepenumero vsu venire potest, vt hoc pacto neutiquam forma integralis simplicissima obtineatur, vel ea demum per plures ambages inde deriuari debeat. Veluti ex hac aequatione:

$$\frac{dx}{x}+\frac{dy}{y}=0$$

 primo

primo elicitur integrale logarithmicum

$$lx + ly = la,$$

vnde quidem statim se prodit algebraicum $xy = a$. Verum ex hac forma

$$\frac{dx}{aa+xx} + \frac{dy}{aa+yy} = 0,$$

integratio solita praebet

$$\text{Ang tang.}\, x + \text{Ang tang}\, y = \text{Const.}$$

vnde non tam facile forma integralis algebraica $\frac{x+y}{aa-xy} = C$ deducitur. Ac proposita hac forma:

$$\frac{dx}{\sqrt{(\alpha+\beta x+\gamma xx)}} + \frac{dy}{\sqrt{(\alpha+\beta y+\gamma yy)}} = 0$$

in genere ne patet quidem, vtrum vtraque pars integralis arcu circulari an logarithmo exprimatur. Interim tamen eius integrale ita algebraice exhiberi potest:

$$CC(x-y)^2 + 2\gamma Cxy + \beta C(x+y) + 2\alpha C + \tfrac{1}{4}\beta\beta - \alpha\gamma = 0$$

quae certe forma simplicissima nonnisi per plures ambages ex integrali transcendente deriuatur.

4. His quidem casibus perspicitur, quomodo reductionem ad formam algebraicam institui oporteat, sed ante aliquot annos eiusmodi integrationes protuli, in quibus ne hoc quidem vllo modo praestari potest. Veluti si proposita sit haec aequatio:

$$\frac{dx}{\sqrt{(1+x^4)}} + \frac{dy}{\sqrt{(1+y^4)}} = 0.$$

integrationem neque per logarithmos neque arcus circu-

circulares expedire licet, vt inde deinceps simili ratione aequatio algebraica colligi posset: interim tamen ostendi huius integrale idque adeo completum hoc modo algebraice exprimi:

$$0 = 2C + (CC - 1)(xx + yy) - 2(1 + CC)xy + 2Cxxyy$$

vbi C denotat constantem per integrationem ingressam. Quin etiam huius aequationis multo latius patentis

$$\frac{dx}{\sqrt{(\alpha + 2\beta x + \delta xx + 2\delta x^3 + \varepsilon x^4)}} + \frac{dy}{\sqrt{(\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4)}} = 0$$

integrale completum est

$$\begin{aligned} 0 = 2\alpha C + \beta\beta - \alpha\gamma &+ 2(\beta C - \alpha\delta)(x + y) + (CC - \alpha\varepsilon)(xx + yy) \\ &+ 2(\gamma C - CC - \alpha\varepsilon - \beta\delta)xy + 2(\delta C - \beta\varepsilon)xy(x + y) \\ &+ (2\varepsilon C + \delta\delta - \gamma\varepsilon)xxyy \end{aligned}$$

denotante C item constantem quantitatem arbitrariam per integrationem inuentam. His igitur casibus perspicuum est separationem variabilium, qua aequationes differentiales sunt praeditae, nihil plane iuuare ad integralia earum forma algebraica contenta eruenda, ex quo merito eiusmodi methodus desideratur, cuius beneficio haec integralia statim ex aequationibus differentialibus inuestigari potuissent, in quo negotio certe omnes ingenii vires tentasse non pigebit.

5. Obseruaui igitur hunc scopum ope multiplicatorum idoneorum obtineri posse, quibus aequationes differentiales multiplicatae ita integrabiles euadant, vt integralia statim algebraice expressa pro-

deant. Quod quo clarius perſpiciatur ab aequatione primum propoſita $\frac{dx}{x}+\frac{dy}{y}=0$ exordiar, quae per xy multiplicata ſtatim praebet $y\,dx+x\,dy=0$, cuius integrale eſt $xy=C$. Hoc ergo modo ſublata ſeparatione aequatio in aliam transformatur, quae integrationem admittit, ex quo intelligitur methodum ope multiplicatorum integrandi id praeſtare, quod a ſeparatione variabilium immediate expectari nequeat. Idem euenit in aequatione $\frac{m\,dx}{x}+\frac{n\,dy}{y}=0$, quae per $x^m y^n$ multiplicata integrale praebet $x^m y^n=C$, dum ex ipſa aequatione propoſita ſtatim ad logarithmos fuiſſet peruentum. Simili modo ſi haec aequatio ſeparata:

$$\frac{dx}{1+xx}+\frac{dy}{1+yy}=0$$

multiplicetur in $\frac{(1+xx)(1+yy)}{(x+y)^2}$, aequatio reſultans,

$$\frac{dx(1+yy)+dy(1+xx)}{(x+y)^2}=0$$

integrationem iam ſponte admittit, praebetque integrata:

$$\frac{-1+xy}{x+y}=\text{Conſt. ſeu } \frac{x+y}{1-xy}=a.$$

Hanc vero aequationem

$$\frac{2\,dx}{1+xx}+\frac{dy}{1+yy}=0$$

multiplicari conuenit in $\frac{(xx+1)^2(1+yy)}{(2xy+xx-1)^2}$, vt prodeat

$$\frac{2\,dx(1+xx)(1+yy)+dy(xx+1)^2}{(2xy+xx-1)^2}=0$$

cuius

cuius integrale reperitur

$$\frac{xxy - 2x - y}{2xy + xx - 1} = \text{Const. seu } \frac{2x + y - xxy}{2xy + xx - 1} = a.$$

6. Contra haec exempla, quidus integralia algebraica fine fubfidio feparationis funt eruta, obiicietur, multiplicatores negotium hoc conficientes ex ipfis integralibus illis tranfcendentibus, ad quae feparatio variabilium immediate perducit effe conclufos, iisque adeo praeftantiam methodi per multiplicatores procedentis neutiquam probari. Cui quidem obiectioni primum refpondeo priora exempla ftatim ab inuentis integrationis principiis fimili modo fuiffe expedita, antequam integratio per logarithmos erat explorata, quae ergo nullum fubfidium eo attuliffe eft cenfenda. Tum vero quamuis concedam, in pofterioribus exemplis integrationem per arcus circulares multiplicatores illos idoneos commode fuppeditaffe, id tamen in ipfa euolutione minus cernitur, eademque integratio fine dubio inueniri potuiffet, antequam conftaret formulae $\frac{dx}{1+xx}$ integrale effe arcum circuli tangenti x refpondentem. Verum aequatio fupra allata

$$\frac{dx}{\sqrt{(1+x^4)}} + \frac{dy}{\sqrt{(1+y^4)}} = 0$$

cuius integrale completum algebraice exhibere licet nulli amplius dubio locum relinquit, cum enim neutrius partis integrale ne conceffis quidem logarithmis vel arcubus circularibus exhiberi poffit, eiusque forma ad genus quantitatum tranfcendentium

 etiam-

etiamnum incognitum ſit referenda, haec certe nullum auxilium ad integrale algebraicum inueniendum attuliſſe cenſeri poteſt. Atque hoc multo magis de aequatione illa latius patente in §. 4. propoſita eſt tenendum, quippe cuius integratio omnino ſingularis ex principiis longe diuerſiſſimis a me eſt eruta.

7. Methodus autem, qua tum ſum vſus, tantopere eſt abſcondita, vt vix vlla via ad eadem integralia perducens patere videatur, et cum ſeparatio variabilium nihil plane eo contuliſſet, vix etiam quicquam ab altera methodo ad multiplicatores adſtricta ſperari poſſe videbatur, propterea quod tum ipſe adhuc in ea opinione verſabar, per multiplicatores nihil praeſtari poſſe, niſi quatenus ſeparatio variabilium eodem manuducat; quandoquidem quaeſtio differentialia tantum primi gradus implicaret. Deinceps autem re diligentius conſiderata perſpexi, quoties aequationis cuiusque differentialis integrale completum exhibere licet, ex eo viciſſim ſemper eiusmodi multiplicatorem elici poſſe, per quem ſi aequatio differentialis multiplicetur, non ſolum fiat integrabilis, ſed etiam integrata id ipſum integrale, quod iam erat cognitum, reproducere debeat; ad hoc autem omnino neceſſe eſt vt integrale completum ſit exploratum, dum ex integralibus particularibus nihil plane pro hoc ſcopo concludere licet. Si enim propoſita ſit aequatio differentialis:

$$P\,dx + Q\,dy = 0,$$

cuius integrale completum vndecunque ſit cognitum, conſta-

constabit id aequatione, quae praeter binas variabiles x et y et quantitates constantes in ipsa aequatione differentiali contentas, insuper quantitatem constantem nouam prorsus ab arbitrio nostro pendentem complectetur. Quae si littera C indicetur, eruatur eius valor ex aequatione integrali, ac reperiatur $C = V$, eritque V certa quaedam functio ipsarum x et y; tum autem hac aequatione differentiata $o = dV$, differentiale dV necessario ita formulam differentialem $Pdx + Qdy$ continere debet, vt sit

$$dV = M(Pdx + Qdy),$$

ex qua forma multiplicator M, ad hoc integrale $C = V$ perducens sponte se offert.

8. Quo haec operatio aliquot exemplis illustretur, sumatur primo haec aequatio

$$\frac{m\,dx}{x} + \frac{n\,dy}{y} = o,$$

cuius integrale completum cum sit $x^m y^n = C$, instituta differentiatione prodit:

$$o = m x^{m-1} y^n dx + n x^m y^{n-1} dy \text{ seu}$$

$$o = x^m y^n \left(\frac{m\,dx}{x} + \frac{n\,dy}{y}\right)$$

vnde patet multiplicatorem ad hoc integrale ducentem esse $x^m y^n$.

Deinde cum huius aequationis

$$\frac{dx}{1 + xx} + \frac{dy}{1 + yy} = o$$

inte-

integrale completum ſit

$$1-xy=C(x+y),$$

valor conſtantis arbitrariae hinc fit $C=\frac{1-xy}{x+y}$, cuius differentiatio praebet

$$0=\frac{-dx(1+yy)-dy(1+xx)}{(x+y)^2} \text{ ſeu}$$

$$0=\frac{(1+xx)(1+yy)}{(x+y)^2}\left(\frac{dx}{1+xx}+\frac{dy}{1+yy}\right)$$

vnde multiplicator quaeſitus eſt $=\frac{(1+xx)(1+yy)}{(x+y)^2}$.

Propoſita porro ſit haec aequatio

$$\frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx)}}+\frac{dy}{\sqrt{(\alpha+2\beta y+\gamma yy)}}=0$$

cuius integrale completum

$$CC(x-y)^2-2C(\alpha+\beta x+\beta y+\gamma xy)+\beta\beta-\alpha\gamma=0$$

dat primo

$$C=\frac{+\alpha+\beta(x+y)+\gamma xy+\sqrt{\left(\alpha\alpha+2\alpha\beta(x+y)\begin{matrix}+\alpha\gamma(xx+yy)+4\beta\gamma xy(x+y)\\+4\beta\beta xy\qquad+\gamma\gamma xxyy\end{matrix}\right)}}{(x-y)^2}$$

ſeu

$$C=\frac{+\alpha+\beta(x+y)+\gamma xy+\sqrt{(\alpha+2\beta x+\gamma xx)(\alpha+2\beta y+\gamma yy)}}{(x-y)^2}$$

vel concinnius:

$$\frac{\beta\beta-\alpha\gamma}{C}=+\alpha+\beta(x+y)+\gamma xy+\sqrt{(\alpha+2\beta x+\gamma xx)(\alpha+2\beta y+\gamma yy)}$$

vnde differentiando fit:

$$0=+dx(\beta+\gamma y)+dy(\beta+\gamma x)+\frac{dx(\beta+\gamma x)\sqrt{(\alpha+2\beta y+\gamma yy)}}{\sqrt{(\alpha+2\beta x+\gamma xx)}}+\frac{dy(\beta+\gamma y)\sqrt{(\alpha+2\beta x+\gamma xx)}}{\sqrt{(\alpha+2\beta y+\gamma yy)}}$$

hinc-

hincque colligitur multiplicator quaesitus:

$$M=(\beta+\gamma x)\sqrt{(\alpha+2\beta y+\gamma yy)}+(\beta+\gamma y)\sqrt{(\alpha+2\beta x+\gamma xx)}.$$

9. Simili modo pro aequatione magis complexa:

$$\frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx+2\delta x^3+\varepsilon x^4)}}+\frac{dy}{\sqrt{(\alpha+2\beta y+\gamma yy+2\delta y^3+\varepsilon y^4)}}=0$$

ex eius integrali completo supra exhibito multiplicator idoneus M inuestigari poterit, ex quo si statim fuisset cognitus idem hoc integrale immediate elici potuisset. Verum hic opus multo maius molior, quod autem primo conatu neutiquam ad finem perducere licebit; ex quo satis mihi equidem praestitisse videbor, si saltem prima quasi lineamenta nouae atque maxime desiderandae methodi adumbra, vero cuius ope, proposita huiusmodi aequatione differentiali multiplicator idoneus eam reddens integrabilem inueniri queat. Ac primo quidem in hoc negotio plurimum obseruasse iuuabit, si vnicus huiusmodi multiplicator innotuerit, ex eo facile infinitos alios idem officium praestantes erui posse. Quodsi enim multiplicator M aequationem differentialem

$$Pdx+Qdy=0$$

integrabilem reddat, ita vt sit

$$\int M(Pdx+Qdy)=V$$

ideoque aequatio integralis $V=C$, quoniam formula

$$dV=M(Pdx+Qdy)$$

per functionem quamcunque quantitatis V multiplicata perinde manet integrabilis, perspicuum est hanc formam $Mf:V$, quaecunque functio ipsius V pro $f:V$ accipiatur semper multiplicatorem idoneum praebere cum sit

$$(Pdx+Qdy)Mf:V=dVf:V$$

ideoque integrabile. Inter infinitos igitur hos multiplicatores idoneos quouis casu eum eligi conueniet, qui negotium facillime conficiat, et integrale si fuerit algebraicum forma simplicissima exhibeat. Etiamsi enim integrale reuera sit algebraicum, omnino fieri potest, vt id ne suspicari quidem liceat, nisi multiplicator idoneus in vsum vocetur, quemadmodum superiora exempla abunde declarant.

10. Sit ergo aequatio differentialis proposita huius formae

$$\frac{dx}{X}+\frac{dy}{Y}=0$$

in qua X sit functio solius x et Y solius y; atque inuestigari oporteat eiusmodi multiplicatorem M, quo illa aequatio algebraice integrabilis reddatur, siquidem fieri potest: quod cum raro eueniat, vicissim assumta multiplicatoris forma M indagasse iuuabit functiones X et Y. Sit primo multiplicator

$$M=\frac{XY}{(\alpha+\beta x+\gamma y)^2},$$

vt integrabilis esse debeat haec forma:

$$\frac{Ydx+Xdy}{(\alpha+\beta x+\gamma y)^2}=0.$$

Hinc

Hinc sumta y constante colligitur integrale

$$\frac{-Y}{\beta(\alpha+\beta x+\gamma y)}+\Gamma:y,$$

sumta autem x constante prodit

$$\frac{-X}{\gamma(\alpha+\beta x+\gamma y)}+\Delta:x,$$

quas ambas formas inter se aequales esse oportet; vnde fit:

$$-\gamma Y+\beta\gamma(\alpha+\beta x+\gamma y)\Gamma:y=-\beta X+\beta\gamma(\alpha+\beta x+\gamma y)\Delta:x$$

seu $\beta X-\gamma Y=\beta\gamma(\alpha+\beta x+\gamma y)(\Delta:x-\Gamma:y)$

sicque patet functiones $\Delta:x$ et $\Gamma:y$ ita comparatas esse debere, vt euoluto posteriori membro termini, qui simul x et y continerent, se mutuo tollant Ex quo intelligitur fore

$$\Delta:x=m\beta x+\text{Const. et } \Gamma:y=m\gamma y+\text{Const.}$$

Statuamus ergo

$$\Delta:x-\Gamma:y=m\beta x-m\gamma y+n, \text{ fietque}$$

$$\beta X-\gamma Y=\beta\gamma\left(\begin{array}{r} m\beta\beta xx-m\gamma\gamma yy+n\beta x+n\gamma y\quad +n\alpha \\ +m\alpha\beta x-m\alpha\gamma y+f \\ -f \end{array}\right\}$$

vnde colligimus:

$$X=\gamma(m\beta\beta xx+\beta(m\alpha+n)x+f+\tfrac{1}{2}n\alpha)$$
$$Y=\beta(m\gamma\gamma yy+\gamma(m\alpha-n)y+f-\tfrac{1}{2}n\alpha)$$

et integralis aequatio algebraica erit

$$m\gamma y-\frac{m\gamma\gamma yy-\gamma(m\alpha-n)y-f+\tfrac{1}{2}n\alpha}{\alpha+\beta x+\gamma y}=\text{Const.}$$

seu $m\beta\gamma xy + n\gamma y - f + \frac{1}{2}n\alpha = C(\alpha + \beta x + \gamma y)$ vel loco C scribendo $C + \frac{1}{2}n$ erit concinnius:

$$m\beta\gamma xy - \tfrac{1}{2}n\beta x + \tfrac{1}{2}n\gamma y - f = C(\alpha + \beta x + \gamma y).$$

11. Videamus iam sub quibus conditionibus haec forma aequationis generalis ista ratione integrabilis euadat:

$$\frac{b\,dx}{Axx + Bx + C} + \frac{k\,dy}{Dyy + Ey + F} = 0.$$

Comparatione ergo cum valoribus inuentis instituta colligitur:

$$\begin{array}{ll} A = bm\beta\beta\gamma & D = km\beta\gamma\gamma \\ B = b\beta\gamma(m\alpha + n) & E = k\beta\gamma(m\alpha - n) \\ C = b\gamma(f + \frac{1}{2}n\alpha) & F = k\beta(f - \frac{1}{2}n\alpha). \end{array}$$

Quoniam hic totum negotium ad rationes litterarum reducitur, sumtis pro primis aequalitatibus

$$\beta = Ak \text{ et } \gamma = Db,$$

concluduntur reliquae:

$m = \frac{1}{ADbbkk}$; $\alpha = \frac{Bk + Eb}{2}$; $n = \frac{Bk - Eb}{2ADbbkk}$ et $f = \frac{ACkk + DFbb}{2ADbbkk}$

praeterea vero haec conditio requiritur, vt sit

$$\frac{4AC - BB}{bb} = \frac{4DF - EE}{kk}$$

quae si habuerit locum, multiplicator idoneus erit

$$M = \frac{(Axx + Bx + Cx)(Dyy + Ey + F)}{bk(\frac{1}{2}(Bk + Eb) + Akx + Dby)^2}$$

et

et aequatio integralis inde resultans erit per bk multiplicando

$$xy - \frac{(Bk - Eb)x}{4Db} + \frac{(Bk - Eb)y}{4Ak} - \frac{ACkk - DFbb}{2ADbk} =$$
$$G(\tfrac{1}{2}(Bk + Eb) + Akx + Dby)$$

quae immutata constante arbitraria G ad hanc formam reuocatur:

$$(x + \tfrac{B}{2A} - GDb)(y + \tfrac{E}{2D} - GAk) = GGADbk$$
$$+ \frac{(4AC - BB)kk + (4DF - EE)bb}{4ADbk}$$

seu $$(\tfrac{2Ax + B}{b} + G)(\tfrac{2Dy + E}{k} + G) = GG + \frac{4AC - BB}{2bb}$$
$$+ \frac{4DF - EE}{2kk}.$$

12. En ergo Theorema minime spernendum, etiamsi eius veritas ex aliis principiis satis manifesta esse queat.

Si haec aequatio differentialis:

$$\frac{b\,dx}{Axx + Bx + C} + \frac{k\,dy}{Dyy + Ey + F} = 0$$

ita fuerit comparata vt sit

$$\frac{4AC - BB}{bb} = \frac{4DF - EE}{kk}$$

tum eius integrale completum erit algebraicum, atque hac aequatione expressum:

$$(\tfrac{2Ax + B}{b})(\tfrac{2Dy + E}{k}) + G(\tfrac{2Ax + B}{b} + \tfrac{2Dy + E}{k}) =$$
$$\frac{4AC - BB}{2bb} + \frac{4DF - EE}{2kk}$$

vbi G denotat constantem arbitrariam per integra-

 tionem

tionem inuectam. Hoc vero integrale inuenitur si aequatio propofita ducatur in hunc multiplicato rem :

$$\frac{(Axx+Bx+C)(Dyy+Ey+F)}{\left(\frac{2Ax+B}{b}+\frac{2Dy+E}{k}\right)^2}.$$

13. Quemadmodum multiplicatori M tribuimus formam

$$\frac{XY}{(\alpha+\beta x+\gamma y)^2},$$

ita etiam formis magis complicatis vti licebit, quod quidem in genere praeftari nequit. Euoluamus autem multiplicatorem

$$M=\frac{YX}{(\alpha+\beta x+\gamma y+\delta xy)^2}$$

vt haec aequatio integrabilis fit efficienda

$$\frac{Ydx+Xdy}{(\alpha+\beta x+\gamma y+\delta xy)^2}=0$$

cuius integratio ad hanc perducit aequationem

$$\frac{-Y}{(\beta+\delta y)(\alpha+\beta x+\gamma y+\delta xy)}+\Gamma:y=\frac{-X}{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)}+\Delta:x$$

quae transformatur in hanc

$$\frac{X}{\gamma+\delta x}-\frac{Y}{\beta+\delta y}=(\alpha+\beta x+\gamma y+\delta xy)(\Delta:x-\Gamma:y)$$

vbi euidens eft ftatui debere

$$\Delta:x=\frac{\zeta x+\eta}{\gamma+\delta x} \text{ et } \Gamma:y=\frac{\zeta y+\theta}{\beta+\delta y}$$

vt nulli termini occurrant qui vtramque variabilem fimul complectantur : hinc ergo fiet

$$\frac{X}{\gamma+\delta x}-\frac{Y}{\beta+\delta y}=\underset{+f}{\eta y}+\frac{(\alpha+\beta x)(\zeta x+\eta)}{\gamma+\delta x}\underset{-f}{-\theta x}-\frac{(\alpha+\gamma y)(\zeta y+\theta)}{\beta+\delta y}$$

vnde

vnde concludimus:

$$X = (\alpha + \beta x)(\zeta x + \eta) - (\gamma + \delta x)(\theta x + f)$$
$$Y = (\alpha + \gamma y)(\zeta y + \theta) - (\beta + \delta y)(\eta y + f)$$

fiue euoluendo:

$$X = (\beta\zeta - \delta\theta)xx + (\alpha\zeta + \beta\eta - \gamma\theta - \delta f)x + \alpha\eta - \gamma f$$
$$Y = (\gamma\zeta - \delta\eta)yy + (\alpha\zeta + \gamma\theta - \beta\eta - \delta f)y + \alpha\theta - \beta f$$

et aequatio integralis erit

$$\frac{\zeta x + \eta}{\gamma + \delta x} - \frac{X}{(\gamma + \delta x)(\alpha + \beta x + \gamma y + \delta x y)} = \text{Conft.}$$

quae loco X fubftituto valore inuento abit in hanc formam

$$\frac{\zeta x y + \eta y + \theta x + f}{\alpha + \beta x + \gamma y + \delta x y} = \text{Conft.}$$

14. Transferamus haec iterum ad formam

$$\frac{b\,dx}{Axx + Bx + C} + \frac{k\,dy}{Dyy + Ey + F} = 0$$

ac fieri oportet:

$$A = b(\beta\zeta - \delta\theta) \qquad D = k(\gamma\zeta - \delta\eta)$$
$$B = b(\alpha\zeta + \beta\eta - \gamma\theta - \delta f) \qquad E = k(\alpha\zeta + \gamma\theta - \beta\eta - \delta f)$$
$$C = b(\alpha\eta - \gamma f) \qquad F = k(\alpha\theta - \beta f).$$

Primae aequationes praebent

$$\theta = \frac{\beta\zeta}{\delta} - \frac{A}{\delta b};\quad \eta = \frac{\gamma\zeta}{\delta} - \frac{D}{\delta k}$$

fecundae vero

$$f = \frac{\alpha\zeta}{\delta} - \frac{Bk - Eb}{2\delta b k} \quad \text{et} \quad \delta = \frac{2A\gamma k - 2D\beta b}{Bk - Eb}$$

vnde ex tertiis colligitur

$$\frac{2Ck(A\gamma k - D\beta b)}{Bk - Eb} = \frac{\gamma}{2}(Bk + Eb) - D\alpha b$$
$$\frac{2Fb(A\gamma k - D\beta b)}{Bk - Eb} = \frac{\beta}{2}(Bk + Eb) - A\alpha k.$$

Hinc

Hinc α elidendo fit :

$$\frac{2(ACkk-DFbb)(Ak\gamma-Db\beta)}{Bk-Eb} = \tfrac{1}{2}(Ak\gamma-Db\beta)(Bk+Eb)$$

vnde cum esse nequeat

$$Ak\gamma - Db\beta = 0,$$

quia alioquin fieret $\delta = 0$, et quantitates θ, η, f infinitae, tum vero quod praecipue est notandum, aequatio integralis prodiret Const. $=$ Const. quo ergo casu nihil indicaretur ; necesse est vt sit

$$4(ACkk-DFbb) = BBkk-EEbb \text{ seu}$$
$$\frac{4AC-BB}{bb} = \frac{4DF-EE}{kk} \text{ vt ante.}$$

Quod autem hic maxime animaduerti meretur, est, quod etsi tres litterae β, γ et ζ manent indefinitae, aequatio tamen integralis a praecedente nonnisi quantitate constante discrepat prodit enim

$$\frac{2\zeta bk}{Bk-Eb} + \frac{k(2Ax+B)+b(2Dy+E)}{2(Ak\gamma-Db\beta)xy+(Bk-Eb)(\beta x+\gamma y)+2(Ck\beta-Fb\gamma)} = \text{Const.}$$

seu $$\frac{\gamma ky(2Ax+B)+\beta k(Bx+2C)-\beta bx(2Dy+E)-\gamma b(Ey+2F)}{k(2Ax+B)+b(2Dy+E)} = \text{Const.}$$

quae forma quomodocunque accipiantur litterae β et γ semper veram aequationem integralem exhibet. Quod cum minus sit perspicuum, ostendisse sufficiet ambas partes β et γ inuoluentes seorsim sumtas eandem relationem inter x et y definire. Constitutis enim his duabus aequationibus :

$$\frac{2Akxy+Bky-Eby-2Fb}{2Akx+2Dby+Bk+Eb} = \text{Const.}$$
$$\frac{2Dbxy-Ebx+Bkx+2Ck}{2Akx+2Dby+Bk+Eb} = \text{Const.}$$

multi-

multiplicetur prior per Db posterior per Ak, fietque summa:

$$\frac{Ak(Bk-Eb)x+Db(Bk-Eb)y+2ACkk-2DFbb}{2Akx+2Dby+Bk+Eb}$$

cuius valor vtique est constans $=\frac{Bk-Eb}{2}$, propterea quod

$$\frac{2ACkk-2DFbb}{Bk+Eb}=\frac{Bk-Eb}{2}$$

vnde patet propositum.

15. Progredior nunc ad formam aequationum magis arduam, quae sit

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$$

sitque multiplicator eam reddens integrabilem

$$M=P\sqrt{X}+Q\sqrt{Y},$$

ita vt aequatio integrationem admittens sit

$$Pdx+Qdy+\frac{Qdx\sqrt{Y}}{\sqrt{X}}+\frac{Pdy\sqrt{X}}{\sqrt{Y}}=0$$

cuius vtrumque membrum seorsim integrabile sit oportet. Pro priore ergo erit $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, posterioris vero integrale statuatur $2V\sqrt{XY}$, vnde colligitur:

$$Q=2X\left(\frac{dV}{dx}\right)+V.\frac{dX}{dx} \text{ et}$$
$$P=2Y\left(\frac{dV}{dy}\right)+V.\frac{dY}{dy}$$

et ob priorem conditionem

$$2Y\left(\frac{ddV}{dy^2}\right)+\frac{3dY}{dy}\left(\frac{dV}{dy}\right)+V.\frac{ddY}{dy^2}=2X\left(\frac{ddV}{dx^2}\right)$$
$$+\frac{3dX}{dx}\left(\frac{dV}{dx}\right)+V.\frac{ddX}{dx^2}$$

ex qua aequatione, si loco V sumserimus certam functionem ipsarum x et y, dispiciendum est, quomodo idonei valores pro functionibus X et Y obtineantur.

16. Demus primo ipsi V valorem constantem puta $V=1$, ac peruenimus ad hanc conditionem:

$$\frac{ddY}{dy^2}=\frac{ddX}{dx^2}.$$

quae aequalitas subsistere nequit, nisi vtrumque membrum seorsim aequetur quantitati constanti, quae sit $=2a$ vnde colligemus

$$X=axx+bx+c \text{ et } Y=ayy+dy+e$$

hincque porro

$$P=\frac{dY}{dy}=2ay+d \text{ et } Q=\frac{dX}{dx}=2ax+b$$

vnde aequatio integralis completa colligitur:

$$2axy+dx+by+2\sqrt{XY}=\text{Const.}$$

Quocica ista aequatio differentialis

$$\frac{dx}{\sqrt{(axx+bx+c)}}+\frac{dy}{\sqrt{(ayy+dy+e)}}=0$$

integrabilis redditur ope multiplicatoris

$$M=(2ay+d)\sqrt{(axx+bx+c)}+(2ax+b)\sqrt{(ayy+dy+e)}$$

ac tum integrale completum reperietur:

$$2axy+dx+by+2\sqrt{(axx+bx+c)(ayy+dy+e)}=C$$

seu

seu sublata irrationalitate:

$$CC-2C(2axy+dx+by)=(4ae-dd)xx+(4ac-bb)yy$$
$$+4bex+4cdy+4ce.$$

Haec autem aequatio differentialis multo latius patet illa quam initio §. 3. attuleram.

17. Tribuamus nunc ipsi V hunc valorem

$$V=\frac{1}{(\alpha+\beta x+\gamma y)^2},$$

si enim loco exponentis 2 indefinitum sumsissem mox patuisset, hanc potestatem accipi debuisse. Erit ergo

$$\left(\frac{dV}{dx}\right)=\frac{-2\beta}{(\alpha+\beta x+\gamma y)^3};\ \left(\frac{dV}{dy}\right)=\frac{-2\gamma}{(\alpha+\beta x+\gamma y)^3}$$
$$\left(\frac{ddV}{dx^2}\right)=\frac{6\beta\beta}{(\alpha+\beta x+\gamma y)^4}\ \text{et}\ \left(\frac{ddV}{dy^2}\right)=\frac{6\gamma\gamma}{(\alpha+\beta x+\gamma y)^4}.$$

His autem valoribus substitutis sequentes oriuntur binae formae

$$12\beta\beta X-\frac{6\beta dX}{dx}(\alpha+\beta x+\gamma y)+\frac{ddX}{dx^2}(\alpha+\beta x+\gamma y)^2=$$
$$12\gamma\gamma Y-\frac{6\gamma dY}{dy}(\alpha+\beta x+\gamma y)+\frac{ddY}{dy^2}(\alpha+\beta x+\gamma y)^2$$

quia igitur in priore y in altera x non vltra duas dimensiones assurgit, euidens est in formulis

$$\frac{ddX}{dx^2}\ \text{et}\ \frac{ddY}{dy^2}$$

variabiles x et y totidem dimensiones habere debere, quia alioquin termini ex x et y mixti vtrinque aequales fieri non possent. Cum ergo ipsae functiones X et Y ad quartum gradum sint ascen-

 surae

furae ponamus

$$X = Ax^4 + 2Bx^3 + Cx^2 + 2Dx + E \text{ et}$$

$$Y = \mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E}.$$

Facta iam substitutione pro priori parte prodit

$$\begin{array}{lllll}
12\beta\beta Ax^4 & +24\beta\beta Bx^3 & +12\beta\beta Cxx & +24\beta\beta Dx & +12\beta\beta E \\
-24\beta\beta A & -36\beta\beta B & -12\beta\beta C & -12\beta\beta D & -12\alpha\beta D \\
+12\beta\beta A & -24\alpha\beta A & -36\alpha\beta B & -12\alpha\beta C & +2\alpha\alpha C \\
 & +12\beta\beta B & +2\beta\beta C & +4\alpha\beta C & \\
 & +24\alpha\beta A & +24\alpha\beta B & +12\alpha\alpha B & \\
 & & +12\alpha\alpha A & &
\end{array}$$

$$\begin{array}{llll}
-24\beta\gamma Ax^3y & -36\beta\gamma Bx^2y & -12\beta\gamma Cxy & -12\beta\gamma Dy \\
+24\beta\gamma A & +24\beta\gamma B & +4\beta\gamma C & +4\alpha\gamma C \\
 & +24\alpha\gamma A & +24\alpha\gamma B &
\end{array}$$

$$+12\gamma\gamma Axxyy + 12\gamma\gamma Bxyy + 2\gamma\gamma Cyy$$

qui termini in ordinem disponantur:

$$\begin{aligned}
&12\gamma\gamma Axxyy + 12\gamma\gamma Bxyy + 12\gamma(2\alpha A - \beta B)xxy \\
&+2\gamma\gamma Cyy + 8\gamma(3\alpha B - \beta C)xy + 2(6\alpha\alpha A - 6\alpha\beta B + \beta\beta C)xx \\
&+4\gamma(\alpha C - 3\beta D)y + 4(3\alpha\alpha B - 2\alpha\beta C + 3\beta\beta D)x \\
&+2(\alpha\alpha C - 6\alpha\beta D + 6\beta\beta E)
\end{aligned}$$

Simili vero modo altera pars erit

$$\begin{aligned}
&12\delta\delta\mathfrak{A}xxyy + 12\delta\delta\mathfrak{B}xxy + 12\delta(2\alpha\mathfrak{A} - \gamma\mathfrak{B})xyy + 2\delta\delta\mathfrak{C}xx \\
&+8\delta(3\alpha\mathfrak{B} - \gamma\mathfrak{C})xy + 2(6\alpha\alpha\mathfrak{A} - 6\alpha\gamma\mathfrak{B} + \gamma\gamma\mathfrak{C})yy \\
&+4\delta(\alpha\mathfrak{C} - 3\gamma\mathfrak{D})x + 4(3\alpha\alpha\mathfrak{B} - 2\alpha\gamma\mathfrak{C} + 3\gamma\gamma\mathfrak{D})y \\
&+2(\alpha\alpha\mathfrak{C} - 6\alpha\gamma\mathfrak{D} + 6\gamma\gamma\mathfrak{E}).
\end{aligned}$$

18. Coaequentur nunc inter se termini homologi vtriusque formae, et sequentibus aequationibus erit satis faciendum

$xxyy$	$\gamma\gamma A = \mathfrak{G}\mathfrak{G}\mathfrak{A}$
xxy	$2\alpha\gamma A - \mathfrak{G}\gamma B = \mathfrak{G}\mathfrak{G}\mathfrak{B}$
xyy	$\gamma\gamma B = 2\alpha\mathfrak{G}\mathfrak{A} - \mathfrak{G}\gamma\mathfrak{B}$
xx	$6\alpha\alpha A - 6\alpha\mathfrak{G}B + \mathfrak{G}\mathfrak{G}C = \mathfrak{G}\mathfrak{G}\mathfrak{C}$
yy	$\gamma\gamma C = 6\alpha\alpha\mathfrak{A} - 6\alpha\gamma\mathfrak{B} + \gamma\gamma\mathfrak{C}$
xy	$3\alpha\gamma B - \mathfrak{G}\gamma C = 3\alpha\mathfrak{G}\mathfrak{B} - \mathfrak{G}\gamma\mathfrak{C}$
x	$3\alpha\alpha B - 2\alpha\mathfrak{G}C + 3\mathfrak{G}\mathfrak{G}D = \alpha\mathfrak{G}\mathfrak{C} - 3\mathfrak{G}\gamma\mathfrak{D}$
y	$\alpha\gamma C - 3\mathfrak{G}\gamma D = 3\alpha\alpha\mathfrak{B} - 2\alpha\gamma\mathfrak{C} + 3\gamma\gamma\mathfrak{D}$
1	$\alpha\alpha C - 6\alpha\mathfrak{G}D + 6\mathfrak{G}\mathfrak{G}E = \alpha\alpha\mathfrak{C} - 6\alpha\gamma\mathfrak{D} + 6\gamma\gamma\mathfrak{E}.$

Tres autem primae aequationes tantum duas dant determinationes:

$$\mathfrak{G} = \frac{2\alpha A\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}} + B\sqrt{A}} \text{ et } \gamma = \frac{2\alpha\mathfrak{A}\sqrt{A}}{B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A}}$$

quarta et quinta itidem vnicam determinationem suppeditant:

$$C - \mathfrak{C} = \frac{3(\mathfrak{A}BB - A\mathfrak{B}\mathfrak{B})}{2A\mathfrak{A}} = \tfrac{3}{2}\left(\frac{BB}{A} - \frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}\right)$$

quae eadem quoque ex sexta sequitur. Satuatur ergo

$$C = \frac{3BB}{2A} + n \text{ et } \mathfrak{C} = \frac{3\mathfrak{B}\mathfrak{B}}{2\mathfrak{A}} + n$$

ſeptima et octaua etiam vnicam determinationem inuoluunt

$$\frac{D\sqrt{A}+\mathfrak{D}\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}}=\frac{A\mathfrak{B}\mathfrak{B}+\mathfrak{A}BB-B\mathfrak{B}\sqrt{A\mathfrak{A}}+2nA\mathfrak{A}}{4A\mathfrak{A}\sqrt{A\mathfrak{A}}}$$ vel

$$D\sqrt{A}+\mathfrak{D}\sqrt{\mathfrak{A}}=\frac{B^3}{4A\sqrt{A}}+\frac{\mathfrak{B}^3}{4\mathfrak{A}\sqrt{\mathfrak{A}}}+\frac{nB}{2\sqrt{A}}+\frac{n\mathfrak{B}}{2\sqrt{\mathfrak{A}}}$$

ſtatuatur ergo

$$D=\frac{B^3}{4AA}+\frac{nB}{2A}+\frac{m}{2\sqrt{A}} \quad \text{et} \quad \mathfrak{D}=\frac{\mathfrak{B}^3}{4\mathfrak{A}\mathfrak{A}}+\frac{n\mathfrak{B}}{2\mathfrak{A}}-\frac{m}{2\sqrt{\mathfrak{A}}}$$

qui valores in vltima aequatione ſubſtituti praebent:

$$24(AE-\mathfrak{A}\mathfrak{E})=\frac{3B^4}{2AA}+\frac{6nBB}{A}+\frac{12mB}{\sqrt{A}}$$
$$-\frac{3\mathfrak{B}^4}{2\mathfrak{A}\mathfrak{A}}-\frac{6n\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}+\frac{12m\mathfrak{B}}{\sqrt{\mathfrak{A}}}$$

quare commode ſtatui licebit

$$E=\frac{B^4}{16A^3}+\frac{nBB}{4AA}+\frac{mB}{2A\sqrt{A}}+\frac{l}{A}$$
$$\mathfrak{E}=\frac{\mathfrak{B}^4}{16\mathfrak{A}^3}+\frac{n\mathfrak{B}\mathfrak{B}}{4\mathfrak{A}\mathfrak{A}}-\frac{m\mathfrak{B}}{2\mathfrak{A}\sqrt{\mathfrak{A}}}+\frac{l}{\mathfrak{A}}.$$

19. Cum autem ſumſerimus $V=\frac{1}{(\alpha+\beta x+\gamma y)^4}$ erit

$$Q=\frac{-4\beta(Ax^4+2Bx^3+Cxx+2Dx+E)}{(\alpha+\beta x+\gamma y)^5}+\frac{2(2Ax^3+3Bxx+Cx+D)}{(\alpha+\beta x+\gamma y)^4}$$
$$P=\frac{-4\gamma(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}{(\alpha+\beta x+\gamma y)^5}+\frac{2(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})}{(\alpha+\beta x+\gamma y)^4}$$

ſiue

$$Q=\frac{2\gamma y(2Ax^3+3Bxx+Cx+D)+2(2\alpha A-\beta B)x^3+2(3\alpha B-\beta C)xx+2(\alpha C-3\beta D)x+2(\alpha D-2\beta E)}{(\alpha+\beta x+\gamma y)^5}$$

$$P=\frac{2\beta x(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})+2(2\alpha\mathfrak{A}-\gamma\mathfrak{B})y^3+2(3\alpha\mathfrak{B}-\gamma\mathfrak{C})yy+2(\alpha\mathfrak{C}-3\gamma\mathfrak{D})y+2(\alpha\mathfrak{D}-2\gamma\mathfrak{E})}{(\alpha+\beta x+\gamma y)^5}$$

vnde inueſtigari oportet integrale formulae $Pdx+Qdy$

ad

ad quod si deinceps addatur $\frac{2\sqrt{XY}}{(\alpha+\beta x+\gamma y)^2}$, aggregatum quantitati constanti aequatum exhibebit integrale completum aequationis

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$$

Pro illo autem integrali inueniendo, ex prioribus valoribus pro P et Q exhibitis, notetur fore separatim

$$\int Q\,dy=\frac{2\beta(Ax^4+Bx^3+Cxx+2Dx+E)}{\gamma(\alpha+\beta x+\gamma y)^2}-\frac{2(2Ax^3+Bxx+Cx+D)}{\gamma(\alpha+\beta x+\gamma y)}+\Gamma:x$$

$$\int P\,dx=\frac{2\gamma(\mathfrak{A}y^4+\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}{\beta(\alpha+\beta x+\gamma y)^2}-\frac{2(2\mathfrak{A}y^3+\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})}{\beta(\alpha+\beta x+\gamma y)}+\Delta:y$$

quae duae expressiones aequales esse debent: quem in finem ponatur

$$\Gamma:x=\frac{2(Axx+Bx+N)}{\beta\gamma}\quad\text{et}\quad\Delta:y=\frac{2(\mathfrak{A}yy+\mathfrak{B}y+\mathfrak{N})}{\beta\gamma}$$

fietque

$\frac{1}{2}\beta\gamma(\alpha+\beta x+\gamma y)^2\int Q\,dy$	$\frac{1}{2}\beta\gamma(\alpha+\beta x+\gamma y)^2\int P\,dx$
$+A\gamma\gamma xxyy$	$+\mathfrak{A}\beta\beta xxyy$
$+B\gamma\gamma xyy$	$+\beta(2\mathfrak{A}\alpha-\mathfrak{B}\gamma)xyy$
$+\gamma(2A\alpha-B\beta)xxy$	$+\mathfrak{B}\beta\beta xxy$
$+N\gamma\gamma yy$	$+(\mathfrak{A}\alpha\alpha-\mathfrak{B}\alpha\gamma+\mathfrak{N}\gamma\gamma)yy$
$+(A\alpha\alpha-B\alpha\beta+N\beta\beta)xx$	$+\mathfrak{N}\beta\beta xx$
$+\gamma(2B\alpha-C\beta+2N\beta)xy$	$+\beta(2\mathfrak{B}\alpha-\mathfrak{C}\gamma+2\mathfrak{N}\gamma)xy$
$+\gamma(2N\alpha-D\beta)y$	$+(\mathfrak{B}\alpha\alpha-\mathfrak{C}\alpha\gamma+\mathfrak{D}\gamma\gamma+2\mathfrak{N}\alpha\gamma)y$
$+(B\alpha\alpha-C\alpha\beta+D\beta\beta+2N\alpha\beta)x$	$+\beta(2\mathfrak{N}\alpha-\mathfrak{D}\gamma)x$
$+E\beta\beta-D\alpha\beta+N\alpha\alpha$	$+\mathfrak{E}\gamma\gamma-\mathfrak{D}\alpha\gamma+\mathfrak{N}\alpha\alpha.$

20. Hae conditiones cum praecedentibus (18) perfecte conueniunt si modo sumatur

$$N = \tfrac{1}{6} C \quad \text{et} \quad \mathfrak{N} = \tfrac{1}{6} \mathfrak{C}.$$

Diuidamus singulos terminos per 6γ, vt prodeat valor formulae

$$\tfrac{1}{6}(\alpha + \beta x + \gamma y)^2 \int Q\, dy,$$

qui substitutis valoribus ante inuentis reperietur:

$$xxyy\sqrt{A\mathfrak{A}} + Bxyy\sqrt{\tfrac{\mathfrak{A}}{A}} + \mathfrak{B}xxy\sqrt{\tfrac{A}{\mathfrak{A}}} + \tfrac{1}{6}Cyy\sqrt{\tfrac{\mathfrak{A}}{A}} + \tfrac{1}{6}\mathfrak{C}xx\sqrt{\tfrac{A}{\mathfrak{A}}}$$

$$+ \left(\tfrac{B\mathfrak{B}}{\sqrt{A\mathfrak{A}}} - \tfrac{2}{3}n\right)xy + \left(\tfrac{BB\mathfrak{B}}{4A\sqrt{A\mathfrak{A}}} - \tfrac{nB}{3A} + \tfrac{n\mathfrak{B}}{6\sqrt{A\mathfrak{A}}} - \tfrac{m}{2\sqrt{A}}\right)y$$

$$+ \left(\tfrac{B\mathfrak{B}\mathfrak{B}}{4\mathfrak{A}\sqrt{A\mathfrak{A}}} - \tfrac{n\mathfrak{B}}{3\mathfrak{A}} + \tfrac{nB}{6\sqrt{A\mathfrak{A}}} + \tfrac{m}{2\sqrt{\mathfrak{A}}}\right)x$$

$$+ \tfrac{BB\mathfrak{B}\mathfrak{B}}{16A\mathfrak{A}\sqrt{A\mathfrak{A}}} + \tfrac{n(B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A})^2}{24A\mathfrak{A}\sqrt{A\mathfrak{A}}} - \tfrac{nB\mathfrak{B}}{4A\mathfrak{A}} + \tfrac{m(B\sqrt{\mathfrak{A}} - \mathfrak{B}\sqrt{A})}{4A\mathfrak{A}} + \tfrac{l}{\sqrt{A\mathfrak{A}}}.$$

Sit haec forma breuitatis gratia $= S$ eritque integrale completum:

$$\frac{S + \sqrt{XY}}{(\alpha + \beta x + \gamma y)^2} = \text{Const. seu}$$

$$S + \sqrt{XY} = \text{Const.}(B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A} + 2Ax\sqrt{\mathfrak{A}} + 2\mathfrak{A}y\sqrt{A})^2$$

quod etiam hac forma concinniori exhiberi potest

$$S + \sqrt{XY} = \text{Const}\left(\tfrac{B}{\sqrt{A}} + \tfrac{\mathfrak{B}}{\sqrt{\mathfrak{A}}} + 2x\sqrt{A} + 2y\sqrt{\mathfrak{A}}\right)^2.$$

Quare dum functiones X et Y conditionibus ante definitis sint praeditae, hoc modo habebitur integrale completum aequationis differentialis

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0.$$

21. Haec inuestigatio aliquanto generalius institui potest tribuendo ipsi V talem valorem $\frac{1}{(\alpha+\beta x+\gamma y+\delta xy)^2}$ quo facilius autem calculi molestias superare queamus obseruo, dummodo variabiles x et y quantitate constante augeantur vel minuantur, cum ad hanc formam reduci $\frac{1}{(a+xy)^2}$ reduci posse: expedito autem calculo restitutio facile instituetur. Considerabo ergo hanc aequationis differentialis formam

$$\frac{dx}{\surd X}+\frac{dy}{\surd Y}=0,$$

quam integrabilem reddi assumo ope multiplicatoris $P\surd X+Q\surd Y$, vt integrari debeat haec formula

$$Pdx+Qdy+\frac{Qdx\surd Y}{\surd X}+\frac{Pdy\surd X}{\surd Y}=0.$$

Statuatur partis posterioris integrale $=2V\surd XY$ fietque vt vidimus:

$$Q=2X\left(\frac{dV}{dx}\right)+V.\frac{dX}{dx} \text{ et } P=2Y\left(\frac{dV}{dy}\right)+V.\frac{dY}{dy}.$$

Sit igitur $V=\frac{1}{(a+xy)^2}$, ideoque

$$\left(\frac{dV}{dx}\right)=\frac{-2y}{(a+xy)^3} \text{ et } \left(\frac{dV}{dy}\right)=\frac{-2x}{(a+xy)^3}$$

ita vt habeamus:

$$Q=\frac{-4Xy}{(a+xy)^3}+\frac{dX}{dx}.\frac{1}{(a+xy)^2} \text{ et}$$
$$P=\frac{-4Yx}{(a+xy)^3}+\frac{dY}{dy}.\frac{1}{(a+xy)^2}.$$

Nunc autem effici debet vt formula $Pdx+Qdy$ integrationem admittat, hunc in finem duplici

 modo

modo eius integrale capiatur dum vel y vel x conſtans accipitur, ſicque obtinebimus:

$$\int P\,dx = \frac{4Y}{yy(a+xy)} - \frac{2aY}{yy(a+xy)^2} - \frac{dY}{ydy}\cdot\frac{1}{a+xy} + \frac{\Gamma:y}{yy}$$

$$\int Q\,dy = \frac{4X}{xx(a+xy)} - \frac{2aX}{xx(a+xy)^2} - \frac{dX}{xdx}\cdot\frac{1}{a+xy} + \frac{\Delta:x}{xx}$$

quas duas formas inter ſe aequales reddi oportet. Multiplicando ergo per $xxyy(a+xy)^2$ habebimus:

$$4xxY(a+xy) - 2axxY - \frac{xxydY}{dy}(a+xy) + xx\Gamma:y.(a+xy)^2 =$$

$$4yyX(a+xy) - 2ayyX - \frac{xyydX}{dx}(a+xy) + yy\Delta:x.(a+xy)^2$$

vnde fingamus

$$X = Ax^4 + 2Bx^3 + Cxx + 2Dx + E;\quad \Delta:x = Lxx + Mx + N$$

$$Y = \mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E};\quad \Gamma:y = \mathfrak{L}yy + \mathfrak{M}y + \mathfrak{N}$$

$$\frac{dX}{dx} = 4Ax^3 + 6Bxx + 2Cx + 2D \text{ et}$$

$$\frac{dY}{dy} = 4\mathfrak{A}y^3 + 6\mathfrak{B}yy + 2\mathfrak{C}y + 2\mathfrak{D}$$

Hinc noſtrae expreſſiones induent has formas

$xxyy(a+xy)^2\int Q\,dy$	$xxyy(a+xy)^2\int P\,dx$
$+Lx^4y^4$	$+\mathfrak{L}x^4y^4$
$+Mx^3y^4$	$+2\mathfrak{B}x^3y^4$
$+2Bx^4y^3$	$+\mathfrak{M}x^4y^3$
$+Nxxy^4$	$-2a\mathfrak{A}xxy^4$
$+2(C+aL)x^3y^3$	$+2(\mathfrak{C}+a\mathfrak{L})x^3y^3$
$-2aAx^4y^2$	$+\mathfrak{N}x^4y^2$
$+2(3D+aM)xxy^3$	$-2a\mathfrak{B}xxy^3$
$-2aBx^3y^2$	$+2(3\mathfrak{D}+a\mathfrak{M})x^3y^2$
$+aaLxxyy$	$+aa\mathfrak{L}xxyy$

+ 2

$$\begin{array}{l|l}
+2(2E+a\mathrm{N})xy^2 & +0xy^2 \\
+0x^2y & +2(2\mathfrak{E}+a\mathfrak{N})x^2y \\
+(2aD+aaM)xyy & +0xyy \\
+0xxy & +(2a\mathfrak{D}+aa\mathfrak{M})xxy \\
+(2aE+aaN)yy & +0yy \\
+0xx & +(2a\mathfrak{E}+aa\mathfrak{N})xx.
\end{array}$$

22. Harum formarum coaequatio suppeditat sequentes determinationes:

$\mathfrak{L}=L$; $M=2\mathfrak{B}$; $\mathfrak{M}=2B$; $N=-2a\mathfrak{A}$; $\mathfrak{N}=-2aA$

$\mathfrak{C}=C$; $D=-a\mathfrak{B}$; $\mathfrak{D}=-aB$; $E=aa\mathfrak{A}$; $\mathfrak{E}=aaA$

ita vt habeatur haec aequatio differentialis:

$$\frac{dx}{\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}}+\frac{dy}{\sqrt{(\frac{E}{aa}y^4-\frac{2D}{a}y^3+Cyy-2aBy+aaA)}}=0$$

cuius integrale completum est

$$\frac{2Bxxy-\frac{2D}{a}xyy-2aAxx-\frac{2E}{a}yy+2Cxy-2aBx+2Dy+2\sqrt{XY}}{(a+xy)^2}=\text{Const.}$$

Hic obseruo si ponamus $y=\frac{-a}{z}$, prodire aequationem initio allatam:

$$\frac{dx}{\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}}+\frac{dz}{\sqrt{(Az^4+2Bz^3+Czz+2Dz+E)}}=0$$

cuius propterea integrale nunc etiam per principia integrationis maxime naturalia assignari potest, cum antea methodo admodum indirecta eo fuissem deductus. Integrale quippe est

$$Axxzz+Bxz(x+z)+Cxz+D(x+z)+E+G(x-z)^2=\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)(Az^4+2Bz^3+Czz+2Dz+E)}$$

 quae-

quae ab irrationalitate liberata induit hanc formam:

$$GG(x-z)^2+2G(Axxzz+Bxz(x+z)+Cxz+D(x+z)+E)$$
$$+(BB-AC)xxzz-2ADxz(x+z)-AE(x+z)^2-2BDxz$$
$$-2BE(x+z)+DD-CE=0$$

quae aequatio in hanc formam reducta cum superiori conuenit

$$(2AG+BB-AC)xxzz+2(BG-AD)xz(x+z)$$
$$+(GG-AE)(x+z)^2-2(2GG+BD-CG)xz$$
$$+2(DG-BE)(x+z)+2EG+DD-CE=0.$$

23. Si nunc scrutari velimus, sub quibus conditionibus haec aequatio differentialis integrationem admittat

$$\frac{dx}{\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}}+\frac{dy}{\sqrt{(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}}=0$$

concipiamus hanc nasci ex illa ponendo $z=\frac{fy+g}{hy+k}$, ita vt aequatio integralis futura sit

$$(2AG+BB-AC)xx(fy+g)^2+2(BG-AD)x(fy+g)(hxy$$
$$+kx+fy+g)+(GG-AE)(hxy+kx+fy+g)^2-2(2GG$$
$$-CG+BD)x(fy+g)(hy+k)+2(DG-BE)(hy+k)(hxy$$
$$+kx+fy+g)+(2EG+DD-CE)(hy+k)^2=0.$$

At vero coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ ex his quantitatibus f, g, h, k ita definiuntur vt sit

$$\mathfrak{A}(fk-gh)^2=Af^4+2Bf^3h+Cffhh+2Dfh^3+Eh^4$$
$$\mathfrak{B}(fk-gh)^2=2Af^3g+Bff(3gh+fk)+Cfh(fk+gh)$$
$$+Dhh(3fk+gh)+2Eh^3k$$
$$\mathfrak{C}(fk-gh)^2$$

$$\mathfrak{C}(fk-gh)^2 = 6Af^2g^2+6Bfg(fk+gh)+C(fk+gh)^2$$
$$+6Dhk(fk+gh)+5Ehhkk+2Cfghk$$

$$\mathfrak{D}(fk-gh)^2 = 2Afg^3+Bgg(gh+3fk)+Cgk(fk+gh)$$
$$+Dkk(fk+3gh)+2Ehk^3$$

$$\mathfrak{E}(fk-gh)^2 = Ag^4+2Bg^3k+Cggkk+2Dgk^3+Ek^4.$$

24. Videamus autem quousque problema in genere aggressi calculum expedire queamus. Sit igitur proposita aequatio $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, quae per $P\sqrt{X}+Q\sqrt{Y}$ multiplicata fiat integrabilis, sitque integrale

$$\int(Pdx+Qdy)+\frac{2\sqrt{XY}}{(\alpha+\beta x+\gamma y+\delta xy)^2}=\text{Const.}$$

eritque vt vidimus:

$$Q=\frac{-4X(\beta+\delta y)}{(\alpha+\beta x+\gamma y+\delta xy)^3}+\frac{dX}{dx(\alpha+\beta x+\gamma y+\delta xy)^2}$$
$$P=\frac{-4Y(\gamma+\delta x)}{(\alpha+\beta x+\gamma y+\delta xy)^3}+\frac{dY}{dy(\alpha+\beta x+\gamma y+\delta xy)^2}$$

vnde colligimus

$$(\gamma+\delta x)^2(\alpha+\beta x+\gamma y+\delta xy)^2\int Qdy=2(\beta\gamma-\alpha\delta)X$$
$$+(4\delta X-(\gamma+\delta x)\tfrac{dX}{dx})(\alpha+\beta x+\gamma y+\delta xy)$$
$$+(\alpha+\beta x+\gamma y+\delta xy)^2\Delta:x$$

similique modo

$$(\beta+\delta y)^2(\alpha+\beta x+\gamma y+\delta xy)^2\int Pdx=2(\beta\gamma-\alpha\delta)Y$$
$$+(4\delta Y-(\beta+\delta y)\tfrac{dY}{dy})(\alpha+\beta x+\gamma y+\delta xy)$$
$$+(\alpha+\beta x+\gamma y+\delta xy)^2\Gamma:y$$

quae duae formae ad consensum perduci debent, ita vt prima per $(\gamma+\delta x)^2$, altera vero per $(\beta+\delta y)^2$

 diuisa

diuisa eandem functionem exhibeant. Quamobrem necesse est vt prior per $(\gamma+\delta x)^2$, posterior per $(\beta+\delta y)^2$ diuisionem admittat, cui ergo requisito ante omnia est satisfaciendum.

25. Euoluamus priorem valorem partibus ab y pendentibus distinguendis:

$$\text{I. } 2(\beta\gamma-\alpha\delta)X+4\delta(\alpha+\beta x)X-(\alpha+\beta x)(\gamma+\delta x)\tfrac{dX}{dx} + (\alpha+\beta x)^2\Delta:x$$

$$\text{II. } -y(\gamma+\delta x)(4\delta X-(\gamma+\delta x)\tfrac{dX}{dx}+2(\alpha+\beta x)\Delta:x)$$

$$\text{III. } +yy(\gamma+\delta x)^2\Delta:x$$

quae expressio per $(\gamma+\delta x)^2$ diuisibilis esse debet; cum ergo tertia pars sponte sit diuisibilis pro secunda ponamus

$$(\alpha+\beta x)\Delta:x+2\delta X=(\gamma+\delta x)R$$

et prima pars erit

$$2(\beta\gamma-\alpha\delta)X+2\delta(\alpha+\beta x)X+(\alpha+\beta x)(\gamma+\delta x)R - (\alpha+\beta x)(\gamma+\delta x)\tfrac{dX}{dx}$$

quae redit ad hanc formam:

$$(\gamma+\delta x)(2\beta X+(\alpha+\beta x)R-(\alpha+\beta x)\tfrac{dX}{dx})$$

ita vt

$$2\beta X+(\alpha+\beta x)(R-\tfrac{dX}{dx})$$

adhuc diuisionem per $\gamma+\delta x$ admittere debeat. Cui conditioni satisfit sumendo

$$R=\tfrac{\beta}{\delta}\Delta:x-\tfrac{(\alpha+\beta x)}{\delta}\Delta':x+(\gamma+\delta x)S,$$

vnde

vnde fit

$$X = \frac{\beta\gamma - \alpha\delta}{2\delta\delta}\Delta : x - \frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta}\Delta' : x + \frac{(\gamma+\delta x)^2}{2\delta}S$$

ideoque prima pars erit

$$(\gamma+\delta x)^2\left(\frac{\beta}{\delta}R - \frac{(\alpha+\beta x)dR}{2\delta dx}\right) + \tfrac{1}{2}(\alpha+\beta x)(\gamma+\delta x)^2 S$$

siue

$$(\gamma+\delta x)^2\Big(\frac{\beta\beta}{\delta\delta}\Delta : x - \frac{\beta(\alpha+\beta x)}{\delta\delta}\Delta' : x + \frac{(\alpha+\beta x)^2}{2\delta\delta}\Delta'' : x$$
$$+ \frac{\beta(\gamma+\delta x)}{\delta}S - \frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta}\cdot\frac{dS}{dx}\Big)$$

deinde secunda

$$y(\gamma+\delta x)^2\left\{\begin{array}{l}\frac{2\beta}{\delta}\Delta : x - \frac{(\alpha+\beta x)}{\delta}\Delta' : x + \frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta}\Delta'' : x \\ \qquad + (\gamma+\delta x)S - \frac{(\gamma+\delta x)^2}{2\delta}\cdot\frac{dS}{dx}\end{array}\right\}$$

ac tertia

$$yy(\gamma+\delta x)^2\Delta : x.$$

Quocirca formulae

$$(\alpha+\beta x+\gamma y+\delta xy)^2\int Q\,dy$$

valor erit

$$\frac{\beta\beta}{\delta\delta}\Delta : x + \frac{2\beta}{\delta}y\Delta : x + yy\Delta : x - \frac{\beta(\alpha+\beta x)}{\delta\delta}\Delta' : x - \frac{(\alpha+\beta x)}{\delta}y\Delta' : x$$
$$+ \frac{(\alpha+\beta x)^2}{2\delta\delta}\Delta'' : x + \frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta}y\Delta'' : x$$
$$+ \frac{\beta}{\delta}(\gamma+\delta x)S + (\gamma+\delta x)yS - \frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta}\cdot\frac{dS}{dx}$$
$$- \frac{(\gamma+\delta x)^2}{2\delta}y\cdot\frac{dS}{dx}$$

seu ita concinnius expressus:

$$\frac{(\beta+\delta y)^2}{\delta\delta}\Delta : x - \frac{(\alpha+\beta x)(\beta+\delta y)}{\delta\delta}\Delta' : x + \frac{(\alpha+\beta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\Delta'' : x$$
$$+ \frac{(\gamma+\delta x)(\beta+\delta y)}{\delta}S - \frac{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\frac{dS}{dx}$$

cui

cui alter aequalis fieri debet, qui eſt

$$\frac{(\gamma+\delta x)^2}{\delta\delta}\Gamma:y-\frac{(\alpha+\gamma y)(\gamma+\delta x)}{\delta\delta}\Gamma':y+\frac{(\alpha+\gamma y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\Gamma'':y$$
$$+\frac{(\beta+\delta y)(\gamma+\delta x)}{\delta}\mathfrak{S}-\frac{(\beta+\delta y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\frac{d\mathfrak{S}}{dy}$$

26. Quodſi iam ponamus:

$$\Delta:x=\delta\delta(Axx+2Bx+C) \text{ et } S=\delta(Dxx+2Ex+F)$$

item

$$\Gamma:y=\delta\delta(\mathfrak{A}yy+2\mathfrak{B}y+\mathfrak{C}) \text{ et } \mathfrak{S}=\delta(\mathfrak{D}yy+2\mathfrak{E}y+\mathfrak{F})$$

reperientur noſtrae expreſſiones ita euolutae

$(\alpha+\beta x+\gamma y+\delta xy)^2\int Q\,dy$	$(\alpha+\beta x+\gamma y+\delta xy)^2\int P\,dx$
$+\delta\delta Axxyy$	$+\delta\delta\mathfrak{A}xxyy$
$+2\delta\delta Bxyy$	$+\delta(\gamma\mathfrak{A}-\beta\mathfrak{D}+\delta\mathfrak{E})xyy$
$+\delta(\beta A-\gamma D+\delta E)xxy$	$+2\delta\delta\mathfrak{B}xxy$
$+\delta\delta Cyy$	$+\delta(\gamma\mathfrak{E}-\alpha\mathfrak{D})yy$
$+\delta(\beta E-\alpha D)xx$	$+\delta\delta\mathfrak{C}xx$
$+(2\beta\delta B+(\beta\gamma-\alpha\delta)A-\gamma\gamma D+\delta\delta F)xy$	$+(2\gamma\delta\mathfrak{B}+(\beta\gamma-\alpha\delta)\mathfrak{A}-\beta\beta\mathfrak{D}+\delta\delta\mathfrak{F})xy$
$+(\alpha\gamma A-2\alpha\delta B+2\beta\delta C-\gamma\gamma E+\gamma\delta F)y$	$+(\gamma\delta\mathfrak{F}+(\beta\gamma-\alpha\delta)\mathfrak{E}-\alpha\beta\mathfrak{D})y$
$+(\beta\delta F+(\beta\gamma-\alpha\delta)E-\alpha\gamma D)x$	$+(\alpha\beta\mathfrak{A}-2\alpha\delta\mathfrak{B}+2\gamma\delta\mathfrak{C}-\beta\beta\mathfrak{E}+\beta\delta\mathfrak{F})x$
$+\alpha\alpha A-2\alpha\beta B+\beta\beta C-\alpha\gamma E+\beta\gamma F$	$+\alpha\alpha\mathfrak{A}-2\alpha\gamma\mathfrak{B}+\gamma\gamma\mathfrak{C}-\alpha\beta\mathfrak{E}+\beta\gamma\mathfrak{F}$

vnde

vnde nonnisi sequentes sex determinationes deducuntur

$$\mathfrak{A} = A$$

$$\mathfrak{B} = \frac{\beta A - \gamma D}{2\delta} + \tfrac{1}{2} E$$

$$\mathfrak{C} = \frac{\beta E - \alpha D}{\delta}$$

$$\mathfrak{D} = \frac{2\gamma\delta B - \gamma\gamma A - \delta\delta C}{\alpha\delta - \beta\gamma}$$

$$\mathfrak{E} = \frac{2\alpha\delta B - \alpha\gamma A - \beta\beta C}{\alpha\delta - \beta\gamma}$$

$$\mathfrak{F} = F - \frac{\gamma E}{\delta} - \frac{\alpha\beta\gamma A + 2\alpha\beta\delta B - \beta\beta\beta C}{\delta(\alpha\delta - \beta\gamma)}$$

his enim omnibus illis conditionibus satisfit. Sic igitur omnes litterae A, B, C, D, E, F vna cum α, β, γ, δ arbitrio nostro manent relictae, ex quibus deinde colligitur functio

$$2X = \delta\delta D x^4 + 2\delta(\delta E + \gamma D - \beta A)x^3 + (\delta\delta F + 4\gamma\delta E + \gamma\gamma D - 2\beta\delta B - (\beta\gamma + 3\alpha\delta)A)xx$$
$$+ 2(\gamma\delta F + \gamma\gamma E - \alpha\gamma A - 2\alpha\delta B)x + \gamma\gamma F - 2\alpha\gamma B + (\beta\gamma - \alpha\delta)C.$$

27. Hunc autem calculum vlterius non prosequor, cum nunc quidem sufficiat methodum directam et rei naturae conformem aperuisse, quae ad easdem integrationes omnino singulares, quas olim ex longe aliis principiis erueram, perducat. In augmentum igitur huius scientiae plurimum interit istam nouam methodum omni studio penitius scrutari. Hunc in finem adhuc obseruo aliam multiplicatoris formam adhiberi posse, cuius ope talis forma

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

integrabilis reddi queat. Statuatur scilicet multiplicator $M = P + Q\sqrt{XY}$, vt integrabilis esse debeat haec forma

$$\frac{P\,dx}{\sqrt{X}} + Q\,dy\sqrt{X} + \frac{P\,dy}{\sqrt{Y}} + Q\,dx\sqrt{Y} = 0.$$

Fingatur prioris partis integrale $= 2R\sqrt{X}$ posterioris vero $= 2S\sqrt{Y}$, vt integrale completum sit

$$R\sqrt{X} + S\sqrt{Y} = \text{Const.}$$

et facta euolutione reperitur:

$$P = \frac{R\,dX}{dx} + 2X\left(\frac{dR}{dx}\right);\quad P = \frac{S\,dY}{dy} + 2Y\left(\frac{dS}{dy}\right)$$
$$Q = 2\left(\frac{dR}{dy}\right)\quad\ldots\ldots;\quad Q = 2\left(\frac{dS}{dx}\right).$$

Cum igitur debeat esse $\left(\frac{dR}{dy}\right) = \left(\frac{dS}{dx}\right)$, manifestum est formulam $R\,dx + S\,dy$ integrabilem esse debere. Non autem opus est, vt ea algebraicum habeat integrale, sed sufficit vt integrationis charactere sit praedita.

28. Sumatur enim

$$R = \frac{y}{\alpha + \beta xy + \gamma xxyy} \quad\text{et}\quad S = \frac{x}{\alpha + \beta xy + \gamma xxyy}$$

eritque

$$Q = \frac{2\alpha - 2\gamma xxyy}{(\alpha + \beta xy + \gamma xxyy)^2} \quad\text{et}$$
$$P = \frac{y\,dX}{dx(\alpha + \beta xy + \gamma xxyy)} - \frac{2Xyy(\beta + 2\gamma xy)}{(\alpha + \beta xy + \gamma xxyy)^2}$$

simulque

$$P = \frac{x\,dY}{dy(\alpha + \beta xy + \gamma xxyy)} - \frac{2Yxx(\beta + 2\gamma xy)}{(\alpha + \beta xy + \gamma xxyy)^2}$$

ita

ita vt habeatur

$$(\alpha+\beta xy+\gamma xxyy)^2 P=$$

$$\frac{y\,dX}{u}(\alpha+\beta xy+\gamma xxyy)-2yyX(\beta+2\gamma xy)=$$

$$\frac{x\,dY}{dx}(\alpha+\beta xy+\gamma xxyy)-2xxY(\beta+2\gamma xy).$$

Statuatur

$$X=Ax^4+2Bx^3+Cxx+2Dx+E \text{ itemque}$$

$$Y=\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E}$$

ac duo illi valores inter se aequandi postulare deprehenduntur, vt sit

$$\beta=0;\ B=0;\ \mathfrak{B}=0;\ D=0 \text{ et } \mathfrak{D}=0$$

tum vero ii fient:

$$\text{I.}\ =-2\gamma Cx^3y^2+4\alpha Ax^3y-4\gamma Exy^2+2\alpha Cxy$$

$$\text{II.}\ =-2\gamma\mathfrak{C}x^2y^3+4\alpha\mathfrak{A}xy^3-4\gamma\mathfrak{E}x^2y+2\alpha\mathfrak{C}xy$$

vnde colligitur

$$\mathfrak{C}=C;\ \frac{\alpha}{\gamma}=\frac{-\mathfrak{E}}{A}=\frac{-E}{\mathfrak{A}} \text{ seu } \mathfrak{A}\mathfrak{E}=AE.$$

Erit ergo

$$X=Ax^4+Cxx-\frac{\alpha}{\gamma}\mathfrak{A};\ Y=\mathfrak{A}y^4+Cyy-\frac{\alpha}{\gamma}A$$

et aequationis

$$\frac{dx}{\sqrt{(Ax^4+Cxx-\frac{\alpha}{\gamma}\mathfrak{A})}}+\frac{dy}{\sqrt{(\mathfrak{A}y^4+Cyy-\frac{\alpha}{\gamma}A)}}=0$$

integrale completum erit

$$y\sqrt{(Ax^4+Cxx-\tfrac{\alpha}{\gamma}\mathfrak{A})}+x\sqrt{(\mathfrak{A}y^4+Cyy-\tfrac{\alpha}{\gamma}A)}=\text{Const.}(\alpha+\gamma xxyy).$$

29. Ex his exemplis facile intelligitur, fere nouum adhuc analyseos genus desiderari, quo huiusmodi operationes certo ordine institui atque vlterius extendi queant, a quo quidem scopo adhuc longissime sumus remoti. Interim tamen ea, quae hactenus exposui maximi momenti esse videntur, ad vniuersalitatem principii integrandi initio memorati stabiliendam, cum adeo eius beneficio per multiplicatores idoneos eae integrationes quae maxime arduae et cognita principia transcendentes erant visae expediri queant. Mihi quidem cum primum in eas incidissem, nulla alia via eo deducere videbatur praeter eam, qua tum eram vsus; nondum enim animaduerteram semper, quoties cuiuscunque aequationis differentialis integrale completum constaret, ex eo multiplicatorem, quo illa integrabilis reddatur, concludi posse; quae conclusio, si integrale tantum fuisset particulare, neutiquam valuisset. Quamobrem integrationum illarum particularium, quas olim simul ex eodem principio alieno eram consecutus, longe aliter est ratio comparata, neque adhuc perspicere licet; quomodo methodo quadam directa et naturali ad easdem perueniri queat.

30. Eo magis igitur operae erit pretium, indolem harum integrationum particularium accuratius examinari, quod quidem contemplatione casus simplicissimi fiet. Huius igitur aequationis differentialis

$$dx\sqrt{(1+xx)}+dy\sqrt{(1+yy)}+ny\,dx+nx\,dy=0$$

inte-

integrale particulare inueneram esse

$$xx+yy+2xy\sqrt{(1+nn)}=nn$$

similiaque integralia innumerabilia etiam inueni pro eiusmodi aequationibus differentialibus, quae neque a logarithmis neque a circuli quadratura pendent: quare haec aequatio ita spectetur, quasi non per logarithmos integrari posset. Hic igitur primo quaeritur, qua via directa hoc integrale particulare ex forma differentiali concludi queat? deinde quomodo aequatio differentialis comparata esse debeat, vt tale integrale particulare exhiberi queat? Ad has ergo quaestiones primum obseruo, aequationem algebraicam esse integrale completum istius aequationis differentialis:

$$\frac{dx}{\sqrt{(1+xx)}}+\frac{dy}{\sqrt{(1+yy)}}=0,$$

tum vero ex illa sequi

$$x+y\sqrt{(1+nn)}=n\sqrt{(1+yy)} \text{ et}$$
$$y+x\sqrt{(1+nn)}=n\sqrt{(1+xx)}$$

ita vt tam $\sqrt{(1+xx)}$ quam $\sqrt{(1+yy)}$ rationaliter per x et y exprimi queat. Cum igitur hinc fit differentiando

$$\frac{xdx}{\sqrt{(1+xx)}}=\frac{dy+dx\sqrt{(1+nn)}}{n} \text{ et } \frac{ydy}{\sqrt{(1+yy)}}=\frac{dx+dy\sqrt{(1+nn)}}{n}$$

si harum formarum multipla quaecunque ad illam

$$\frac{dx}{\sqrt{(1+xx)}}+\frac{dy}{\sqrt{(1+yy)}}=0$$

addantur, semper prodire aequationem differentialem, cui aequatio algebraica particulariter saltem satisfa-

 ciat

ciat. In genere ergo huius aequationis differentialis

$$\frac{dx+Pxdx}{\sqrt{(1+xx)}}+\frac{dy+Qydy}{\sqrt{(1+yy)}}=\frac{Pdy+Qdx+(Pdx+Qdy)\sqrt{(1+nn)}}{n}$$

integrale particulare erit

$$xx+yy+2xy\sqrt{(1+nn)}=nn.$$

Sit iam $P=x$ et $Q=y$, ac satisfiet huic aequationi

$$dx\sqrt{(1+xx)}+dy\sqrt{(1+yy)}=\frac{xdy+ydx+(xdx+ydy)\sqrt{(1+nn)}}{n}$$

ex integrali vero fit

$$xdx+ydy=-(xdy+ydx)\sqrt{(1+nn)}$$

ita vt habeatur haec aequatio differentialis:

$$dx\sqrt{(1+xx)}+dy\sqrt{(1+yy)}+nxdy+nydx=0$$

cui ergo integrale supra datum particulariter convenit.

31. Transferamus iam haec ad casus latius patentes, et postquam huius aequationis

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$$

inuentum fuerit integrale completum, quod sit $W=$ Const. notetur hinc semper vtrumque valorem radicalem $\sqrt{X}$ et $\sqrt{Y}$ per functiones rationales ipsarum x et y definiri. Sit ergo

$$\sqrt{X}=R \text{ et } \sqrt{Y}=S, \text{ ideoque}$$
$$\frac{dX}{\sqrt{X}}=2dR \text{ et } \frac{dY}{\sqrt{Y}}=2dS$$

Sit iam P functio ipsius x et Q ipsius y; hincque conflatur ista aequatio:

$$\frac{dx+PdX}{\sqrt{X}}+\frac{dy+QdY}{\sqrt{Y}}-2PdR-2QdS=0$$

cui

cui aequatio algebraica $W = $ Conft. certe particulariter fatisfacit. Hinc fi P et Q ita accipiantur, vt formula $P dR + Q dS$ integrationem admittat, cuius integrale fit $= V$, orietur aequatio transfcendens

$$\int \frac{dx + PdX}{\sqrt{X}} + \int \frac{dy + QdY}{\sqrt{Y}} - 2V = \text{Conft.}$$

cui aequationi $W = $ Conft. feu valoribus inde deductis $\sqrt{X} = R$ feu $\sqrt{Y} = S$ particulariter fatisfacit. Tale ergo ratiocinium viam ad huiusmodi integrationes particulares alioquin inuentu difficillimas patefacere videtur.